 submitted 02/09/2021, accepted 16/06/2022, revised 12/07/2022

Relation between understandings of linear algebra concepts in the embodied world and in the symbolic world

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Abstract. For the use of embodied notions in teaching linear algebra, some studies indicate that it is helpful, whereas other studies indicate that it could be problematic or become an obstacle. Hence, additional research is needed. This study is focused on linear (in)dependence and basis, and investigates the relation between their understandings in the embodied and symbolic worlds. We also examine whether students' conceptions in the embodied world can be improved by the instruction emphasizing geometric images, as our previous studies identified some limitations of students' understanding in the embodied world. To address these issues, we designed four tasks aiming to assess students' conceptions of linear (in)dependence, basis, and dimension, and also designed linear algebra lessons emphasizing geometric images of these concepts. These tasks were conducted during the lessons and the data of 38 engineering students was collected. The analysis for the data showed that conceptions in the embodied world was positively associated with conceptions in the symbolic world; however, students' conceptions in the embodied world were not sufficiently improved by the geometric instruction implemented in this study.

Keywords. linear algebra, linear independence, visualization, Tall's model of three worlds, metaphor theory

Résumé. En ce qui concerne l'utilisation de notions incarnées dans l'enseignement de l'algèbre linéaire, certaines études indiquent qu'elle est utile, tandis que d'autres études indiquent qu'elle pourrait être problématique ou devenir un obstacle. Des recherches supplémentaires sont donc nécessaires. Cette étude se concentre sur l'(in)dépendance linéaire et la base, et examine la relation entre leurs compréhensions dans les mondes incarné et symbolique. Nous examinons également si les conceptions des élèves dans le monde incarné peuvent être améliorées par un enseignement mettant l'accent sur les images géométriques, car nos études précédentes ont identifié certaines limites de la compréhension des étudiants dans le monde incarné. Pour aborder ces questions, nous avons conçu quatre tâches visant à évaluer les conceptions des étudiants sur l'(in)dépendance linéaire, la base et la dimension, et nous avons également conçu des leçons d'algèbre linéaire mettant l'accent sur les images géométriques de ces concepts. Ces tâches ont été administrées pendant les leçons et les données de 38 élèves ingénieurs ont été collectées. L'analyse des données a montré que les conceptions dans le monde incarné étaient positivement associées aux conceptions dans le monde symbolique ; cependant, les conceptions des étudiants dans le monde incarné n'ont pas été suffisamment améliorées par l'instruction géométrique mise en œuvre dans cette étude.

Mots-Clés. algèbre linéaire, indépendance linéaire, visualisation, modèle des trois mondes de Tall, théorie des métaphores

1. Introduction

It is widely recognized that linear algebra is a difficult subject to learn because of its abstract and formal nature. Dorier and Sierpinska (2001) stated that "linear algebra remains a cognitively and conceptually difficult subject." Therefore, overcoming the difficulties in teaching linear algebra has been a challenge. Despite many studies in the past two decades, "research results continue to show that students find it difficult to understand its main concepts" (Trigueros & Wawro, 2020). Hence, how to teach linear algebra remains an open research question.

Different approaches to teaching linear algebra have been investigated. A popular approach is to use geometry—namely visual images of geometric vectors which are represented as arrows on a plane or in a space—to teach linear algebra concepts. Actually, many existing linear algebra textbooks use geometry (Harel, 2019). Geometric vectors represented as arrows on a plane or in space, in other words, plane vectors or spatial vectors, can be interpreted as "travels" and therefore they are embodied objects. Vectors represented as arrows are connected to physical senses and it seems that students can easily understand their operations. Hence, it might seem natural to teach linear algebra by using geometry, and such way of teaching might seem to be helpful for students; however, this

This article is an extended version of the paper presented at INDRUM2020 (Kawazoe, 2020). This work was supported by the JSPS KAKENHI Grant Number JP15K12392, JP19H01738.

is not easy. Some researchers identified that visual images help students understand concepts in linear algebra: (Stewart & Thomas, 2007), (Stewart & Thomas, 2009), (Thomas & Stewart, 2011), (Hannah, Stewart, & Thomas, 2013), (Hannah, Stewart, & Thomas, 2014), and (Donevska-Todorova, 2018, p.268). The effectiveness of the use of dynamic geometry software in teaching of linear algebra was also reported: (Gol Tabaghi & Sinclair, 2013), (Dogan, 2018). However, other studies indicated that using geometry may be problematic in teaching linear algebra (Sierpinska, 2000, p.244) and become an obstacle in learning linear algebra concepts (Hillel, 2000). These studies suggest that the use of geometry in teaching of linear algebra should be carefully incorporated with consideration for students' cognition, and that we must be aware that the use of geometry may limit students' views to geometric vectors. Regarding the latter claim, some studies pointed out the importance of balancing the use of geometry in order to avoid students' view to be limited to geometric vectors: e.g., (Gueudet-Chartier, 2004), (Harel, 2017). In addition, there is another report (Bagley & Rabin, 2016) providing evidence for the utility of computational reasoning and questioning the view that computational thinking is mathematically unsophisticated among three modes of thinking: abstract, geometric, and computational. These remarks indicate that the use of geometry in teaching linear algebra and its effectiveness should be additionally investigated.

Despite two separate indications on the use of geometry in teaching linear algebra in the literature, as mentioned above, it seems that both studies are based on a belief that students can understand linear algebra concepts easily in the embodied world, that is, the world of geometric vectors, and discuss only whether students' understanding of linear algebra concepts in the embodied world can be generalized to the understanding of abstract linear algebra concepts. For example, a question discussed by Gueudet-Chartier (2004) was whether it is "possible for students to learn linear algebra by abstracting from geometry," related to the main question "should we teach linear algebra through geometry?", which is the title of the paper. To the best of our knowledge, the belief itself has never been examined deeply in the literature. However, it remains unclear whether we can rely on such a naive belief. We consider that it should be examined before we discuss the effectiveness of the use of geometry in teaching linear algebra. That is a motivation of our research to investigate students' understanding of linear algebra concepts in the context of geometric vectors.

Our previous studies focused on the notion of linear independence and dependence, and we examined students' understanding of this notion in the context of geometric vectors, especially spatial vectors. In a series of our studies, we observed the following results: (1) Many students failed to determine linear dependence of four spatial vectors such that any three of them do not lie on the same plane (Kawazoe, Okamoto, & Takahashi, 2014); (2) Some of those students required a longer time to imagine that three spatial vectors not lying on the same plane span the whole space (Kawazoe & Okamoto, 2016) and (Kawazoe, 2018). Details of these our previous results are described in Section 3. Although our previous studies identified some limitations of students' understanding in the context of geometric vectors, we have not investigated how these observations were related to understanding of concepts and procedures in linear algebra. In this study, we examine the relation between understandings in the context of geometric vectors and in the general context of vectors in the symbolic world, such as, numerical vectors and polynomials.

Therefore, in this study, we focused on concepts of linear (in)dependence and basis, and we investigated the following research questions:

> **RQ1**. Is geometrical understanding of linear (in)dependence in the embodied world related to understanding of linear (in)dependence and basis in the symbolic world?
>
> **RQ2**. Can geometrical understanding of linear (in)dependence in the embodied world, including the case of four spatial vectors, be improved by an instruction emphasizing a geometric image of linear (in)dependence?

The structure of this article is as follows. In Section 2, we describe theoretical frameworks used in our study. Section 3 briefly summarizes of the results of our previous studies. Section 4 presents the context, tasks, and the methodology of this study. In Section 5, we show the results of each task and analyzes the relations between them. In Section 6, we discuss our research questions based on the results of Section 5 and implications of the study regarding the use of geometric images in the teaching of linear algebra.

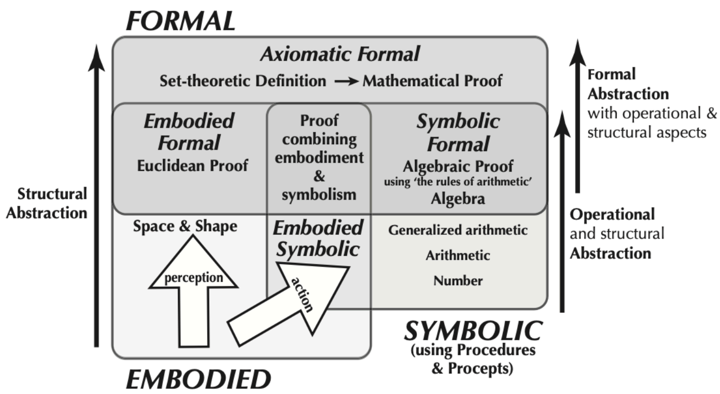
2. Theoretical framework

In our studies, it is important to capture students' comprehension and cognition on linear algebra concepts. For this aim, we use two theories by Tall (2013) and Lakoff and Núñez (2000) as theoretical framework.

2.A. Tall's model of three worlds

In linear algebra, there are various representations of vectors: geometric vectors, numerical vectors, matrices, polynomials, functions, and abstract vectors. Students' conceptual understanding of each of these representations may differ, and the relationship of understanding between the representations is often important in research. In order to capture students' understanding of linear algebra concepts in these different types of representations, Tall's model of three worlds developed by Tall (2013) is useful and used in some studies: e.g., (Stewart & Thomas, 2009), (Stewart & Thomas, 2010), (Stewart, Troup, & Plaxco, 2019), and (Altieri & Schirmer, 2019).

Tall (2013) described the development of mathematical thinking in terms of three worlds: embodied world, symbolic world, and formal world. He proposed these three worlds based on his observation of three different ways in which mathematical thinking can develop: conceptual embodiment, operational symbolic, and axiomatic formalism. Tall (2013) illustrated the outline of the development of the three worlds of mathematics as Figure 1.



(Tall, 2013, p.17, Figure 1.5)

Figure 1: Preliminary outline of the development of the three worlds of mathematics

According to Tall's model, a mathematical concept is a blend of embodiment, symbolism, and formalism. For example, the real number system is a blend of a number line in the embodied world, decimal symbols in the symbolic world, and a complete ordered field $\boldsymbol{R}$ with operations $+$ and $\times$ in the formal world.

> "The whole development of number—from whole number to fraction, to positive and negative numbers, to finite and infinite decimals represented as points on a number line—is a succession of extensional blends, broadening one number system to a larger one with richer properties." (Tall, 2013, p.25)

For higher (university) levels of pure mathematics, Tall stated that "the combination of embodied and symbolic mathematics can be seen as a preliminary stage to the axiomatic formal presentation of mathematics" (Tall, 2013, p.18). In the context of linear algebra, the embodied world is a world of geometric vectors (e.g., arrows), the symbolic world is a world of numerical vectors, matrices, polynomials, and operations represented by symbols, and the formal world is a world of axiomatic vector spaces.

In Tall's view, mathematical thinking starts with physical objects and operations on them, and it is sophisticated by compression of knowledge.

> "The manner in which a process carried out in time may eventually be conceived as a mental concept independent of time is an example of a more general way to think of complicated situations in simple ways.
>
> *Compression of knowledge* occurs when a phenomenon of some kind is conceived in the mind in a simpler or more efficient manner." (Tall, 2013, p.14)

In this study, we use this Tall's view to distinguish students' conception of linear (in)dependence. Let us consider a situation to determine linear independence of given vectors. We can describe how student's approach to the problem differs depending on the level of knowledge compression. In the embodied world, for given geometric vectors, the naivest method is to see whether some vector can be represented as a linear combination of the other vectors by drawing it. When a student becomes to be able to imagine a (partial) set of linear combinations without drawing, he/she can approach this problem by thinking whether some vector is contained in a set of linear combinations of the other vectors. When a student becomes to have a mental image of a space spanned by a non-zero vector, non-parallel two vectors, and three vectors not on the same plane, as a line, a plane, and a space, respectively, he/she can use a more sophisticated method using these geometric images of a spanned space effectively; for example, he/she can intuitively understand that four spatial vectors in a space cannot be linearly independent. In the symbolic world, for given numerical vectors, the naivest method is trying to find coefficients of an equation of a linear dependence heuristically. When a student understands that finding such coefficients is equivalent to finding a solution of a system of linear equations, he/she can use a procedural method, such as the Gaussian elimination. When a student becomes to understand systems of linear equations theoretically, he/she can discuss this problem by looking at the relation between the number of variables and the number of equations, or more sophisticatedly, by looking at the relation between the number of variables and the rank of the coefficient matrix.

As stated in (Tall, 2013), the same idea on development of mathematical thinking was proposed by various other authors. The reader can use one of such theories, such as APOS theory (Arnon et al., 2014), to distinguish students' conceptions.

2.B. Metaphor theory by Lakoff and Núñez

Although Tall's model includes a cognitive viewpoint, in some cases, the metaphor theory by Lakoff and Núñez (2000) gives a deep insight on students' cognitive process of understanding mathematical concepts from a viewpoint of cognitive science. Students' use of metaphors in understanding linear algebra concepts was reported in some studies: e.g., (Plaxco & Wawro, 2015), (Zandieh, Adiredja, & Knapp, 2019). However, the metaphor theory by Lakoff and Núñez (2000) is more fundamental. Lakoff and Núñez (2000) see metaphor to be a central process in all thought of human being, including mathematics, and they proposed that mathematics arises from our embodied experience. They identified various metaphors related to mathematical concepts in (Lakoff & Núñez, 2000). One of the important metaphors they identified is "Basic Metaphor of Infinity (BMI)"; it is explained as follows:

> "We hypothesize that all cases of actual infinity—infinite sets, points at infinity, limits of infinite series, infinite intersections, least upper bounds—are special cases of a single

> general conceptual metaphor in which processes that go on indefinitely are conceptualized as having an end and an ultimate result. We call this metaphor the *Basic Metaphor of Infinity*, or the BMI for short." (Lakoff & Núñez, 2000, p.158)

As described in the next section, we can explain students' cognition of spanned space by using this metaphor.

The metaphor theory can also be applied to the concept of basis of vector space. With the viewpoint of Lakoff and Núñez (2000), space is naturally continuous, but due to Descartes's invention of analytic geometry, it is also conceptualized as "a set of points." Using the word of Lakoff and Núñez (2000), the latter conceptualization is called "discretization" of space. Lakoff and Núñez (2000) identified a central metaphor at the heart of the discretization. The metaphor is called "A Space Is a Set of Points." With this metaphor, a space is regarded as a set of n-tuples of numbers, where a point in the set corresponds to a location in the space. To obtain a location in the space, axes are needed, and each axis is a number line. From the viewpoint of Lakoff and Núñez (2000), discretization in the case of three-dimensional Euclidean space is obtained by a "conceptual blend" of three number lines and the Euclidean space. Discretization of space can be easily applied to vector spaces, because a coordinate system of a vector space is given by a basis of it, where a basis determines axes of the space.

We think that the theory by Lakoff and Núñez (2000) is helpful for designing linear algebra lessons in which students naturally understand concepts by using embodied notions. Therefore, we use this theory in designing linear algebra lessons implemented in this study. In the present study, we use the above two metaphors in designing and implementing linear algebra lessons on spanned space, linear (in)dependence, basis, and dimension in which geometrical viewpoint is emphasized.

3. Results of our previous studies

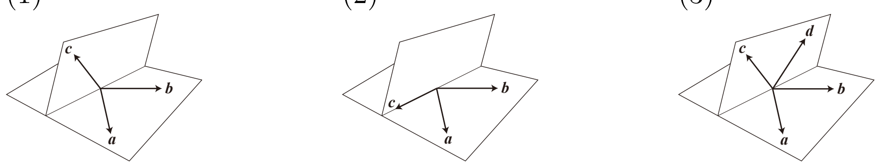
For the convenience of the readers, we briefly summarize our two previous studies (Kawazoe et al., 2014) and (Kawazoe & Okamoto, 2016) closely related to the present study, because the details of (Kawazoe et al., 2014) have not been described in the paper and (Kawazoe & Okamoto, 2016) is written in Japanese.

3.A. Results of the first study

In our first study (Kawazoe et al., 2014), we examined students' understanding of linear independence in the context of geometric vectors by using the following task. In the study, students' understanding of subspace in the context of numerical vector spaces was also examined, but we omit it here to focus on the results related to the present study.

Task A. Determine whether vectors in the following pictures are linearly independent and explain the reasons. Moreover, when they are linearly dependent, give a maximal set of linearly independent vectors.

(1) (2) (3)



Participants of this study were 107 first-year university engineering students who attended a linear algebra course. The test was conducted in linear algebra classes. Results are shown in Table 1.

	correct	incorrect	no answer
(1)	85.05%	5.61%	9.35%
(2)	87.85%	3.74%	8.41%
(3)	62.62%	23.37%	14.02%

Table 1: Results of (1), (2), (3)

The percentage of correct answers for (3) was lower than for (1) and (2). The following example of a student's reasoning for (3) shows a typical error:

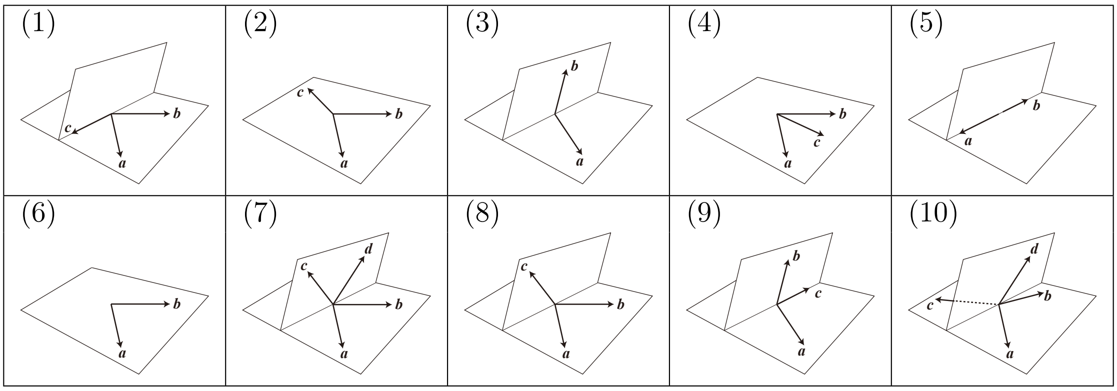
> "As the space generated by three vectors does not contain the other vector, these vectors are linearly independent." (This was originally written in Japanese and translated into English by the author.)

The error indicated that this student failed to understand that three linearly independent spatial vectors span the whole space. However, we could not determine the exact students' difficulties in understanding the case of (3). Hence, a further study was needed to investigate students' understanding of linear (in)dependence in the context of geometric vectors including the case of (3). For this aim, we conducted the second study (Kawazoe & Okamoto, 2016).

3.B. Results of the second study

The second study (Kawazoe & Okamoto, 2016) was designed as a two-round survey. Participants in this study were 71 first-year university engineering students attending a linear algebra course. In the first round, we conducted a paper-based test by using the following task.

Task B. Determine whether spatial vectors given in each picture are linearly independent. Answer "yes" or "no" for each case. Note that each vector lies on a line or a plane shown in the picture. (If there are multiple planes, each vector lies on one of them.)



The test items included the three pictures used in the first study. The test was conducted in the class after students learned the notions of abstract vector space, subspace, spanned space, and linear independence and dependence. The percentages of correct answers for the above question were as follows: (1) 84.5%, (2) 87.3%, (3) 90.1%, (4) 84.5%, (5) 94.4%, (6) 97.2%, (7) 45.1%, (8) 87.3%, (9) 88.7%, (10) 84.5%. Therefore, the percentage of correctness for (7) was significantly lower compared with the others. The picture in (7) is the same as the third picture in Task A in the first study. In this round, we again observed that the case of four spatial vectors, in which any three of them do not lie on the same plane, was quite difficult for students.

In the second round, we conducted a semi-structured interview. This round aimed to investigate why so many students fail to answer (7) correctly. We selected 17 participants among 71 students who attended the first round. The participants were students who had never missed a class until the time the interview was requested, and we distinguished the following two groups: (A) a group of 7 students who answered correctly in all cases, (B) a group of 10 students who failed (7) but answered correctly all the other cases. In the interview, the participants were first asked the following question.

Q1. Explain how you determined linear independency in each question.

If a participant did not use the fact that linear independence can be determined by checking whether some vector can be represented as a linear combination of the others, then the interviewer reminded the participant this fact. Subsequently, all participants were asked the following question.

Q2. What is the geometric figure of the set of all linear combinations of the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in (7)?

For the participants who felt difficulty in imagining the geometric figure asked in Q2, the following question was additionally asked.

Q2'. Why do you feel difficulty in imagining the set of all linear combinations?

In this study, additional two questions were asked for the participants who answered Q2', but we omit them here to focus on the main results. During the interview, the participants looked at the test paper used in the first round, and they answered by drawing diagrams or arrows on paper if necessary. Interviews were video-recorded.

The obtained data were qualitatively analyzed to investigate the difference between Group A and Group B. We observed the following results. No difference was found in the conceptions of linear (in)dependence of two to three geometric vectors, and spanned spaces with one or two dimensions. However, the difference was found in the conceptions of linear (in)dependence of four geometric vectors and of spanned space generated by three linearly independent geometric vectors. Although all students in Group B could answer for Q2 correctly, six of them showed some difficulties. Among them, four students answered that they need a longer time for imagining that three linearly independent geometric vectors span the whole space. It was observed that their imagining processes were common. Their image starts from some shape in a space, then it gradually extends, and finally fills the whole space. There were three types of a starting object. One is a parallelepiped created by three vectors. The second one is a triangular pyramid created by three vectors. And the third one is a plane spanned by two vectors with a cylindrical object along the third vector. Though a starting object is different, those students imagine a space spanned by linearly independent three spatial vectors as a gradually expanding three-dimensional object, which finally fills the whole space.

The second study identified some limitations of students' understanding in the context of geometric vectors. However, it was not investigated how these observations were related to understanding of concepts and procedures in linear algebra. The present study aims to investigate this relation.

3.C. Discussion on the result of the second study from the viewpoint of metaphor theory

As we stated in Section 3.B, we observed students' cognitive process of imagining a space spanned by three linearly independent spatial vectors as a gradually expanding three-dimensional object which finally fills the whole space. This observation can be interpreted by applying the "Basic Metaphor of Infinity (BMI)" stated in Section 2.B.

By applying the BMI to this case, a spanned space in such process can be explained as the "final resultant state" obtained by BMI from an infinite sequence of three-dimensional objects, which is getting larger step by step. In the case of a space spanned by three linearly independent spatial vectors, each three-dimensional object is a subset of the whole space, and difference between the whole space and the object is getting smaller when the object becomes larger. This leads to a cognition that a space spanned by three linearly independent spatial vectors as the final resultant state obtained by BMI coincides with the whole space.

4. Context: the course, students, design of lessons and tasks

We describe the present study starting from this section. The study was conducted in a linear algebra course for engineering students at a Japanese university, but in a special class for students who failed to pass it in their first year. The course consists of a spring semester class and a fall semester class. The former is a 2-credit class, meeting for 90 minutes each week for 15 weeks. The latter is a 4-credit class, meeting for 180 minutes each week for 15 weeks. Each of them is followed by an examination period. The course covers usual linear algebra topics: numerical vector space, matrix, Gaussian elimination, system of linear equations, determinant, etc., in the spring semester; formal vector space, spanned space, linear (in)dependence, basis, dimension, linear map, inner product, orthogonal basis, eigenvalue, eigenvector, and diagonalization, etc., in the fall semester. This study was conducted during the first five weeks in the fall semester. In these weeks, students learned the concepts of formal vector space, spanned space, linear (in)dependence, basis, and dimension.

4.A. Design of lessons

Each lesson included a lecture part and an exercise part. Lectures and exercises were given in the first and second half of the lesson, respectively. In the design of lessons, we regarded the notion of spanned space as a key notion in understanding linear (in)dependence, basis, and dimension. Hence, the instruction of spanned space was carefully implemented so that all students could have a geometric image of it in the embodied world, by considering our observations from the second study (Kawazoe & Okamoto, 2016) and by applying the metaphor theory of Lakoff and Núñez (2000). In order to address the research questions, the lecture part was designed to emphasize geometric images of linear algebra concepts, especially by using the image of a spanned space in the embodied world. In the lecture part, the teacher introduced linear algebra concepts the following way.

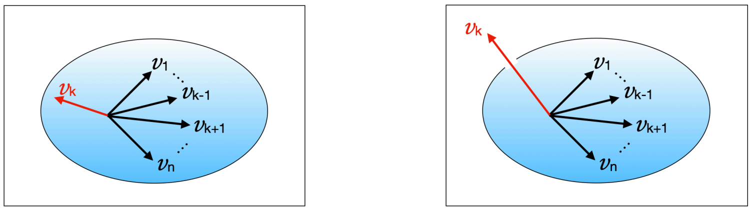
First, the notions of linear combination and spanned space were introduced. A space spanned by three linearly independent spatial vectors was shown to students by using teacher's fingers, and it was emphasized that linear combinations with negative coefficients were contained in the spanned space. In order to make students grasp the correct image of the spanned space, the teacher stressed the importance of imagining a part of the space consisting of linear combinations with some (or all) coefficients being negative and imagining that the figure represented by linear combinations extends infinitely. This instruction was designed based on our finding on students' cognition of spanned space in the previous study (Section 3.B) and its interpretation by using BMI of Lakoff and Núñez (2000) in Section 3.C.

The notions of linear independence and dependence were introduced by using usual formal definitions. The reason was as follows: the students in the class were students who failed to pass the course in the previous years and hence they already learned the notions. Although we used usual formal definitions, the meaning of linear independence and dependence of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ in a vector space were explained in terms of spanned space as follows:

> Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly dependent if and only if one of the n vectors can be represented by a linear combination of the other $n-1$ vectors, that is, one of the n vectors is contained in the space spanned by the other $n-1$ vectors.

> Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly independent if and only if none of the n vectors can be represented by a linear combination of the other $n-1$ vectors, that is, none of the n vectors is contained in the space spanned by the other $n-1$ vectors.

Such explanations emphasized geometric images of the notions. In the above explanations, the teacher used drawings shown in Figure 2. Moreover, it was also explained that linearly independent



(a) (b)

Figure 2: Drawings used in the explanations of (a) linear dependence and (b) independence

vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ give an ascending sequence of vector spaces $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n$, where V_k ($k = 1, 2, \ldots, n$) is the space spanned by $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$. The reason of using such an ascending sequence to explain linear independence is that an iterative process is a basic embodied notion (Lakoff & Núñez, 2000). Subsequently, the notion of basis was introduced by a usual formal definition:

> Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ in a vector space V form a basis of V if and only if they are linearly independent and any vector in V can be represented as their linear combination.

It was explained that the second condition is equivalent to that V is spanned by $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$. In the introduction of basis, the role of basis was explained as to give a coordinate system, and a basis was explained as a set of "axes." Furthermore, it was explained that the second condition means that it contains a sufficient number of axes to represent the whole space, and that the first condition means that there is no extra axis in the set. This instruction was designed based on our view of basis stated in Section 2.B, which is obtained by applying "discretization" of space to vector spaces.

In the exercise part, students worked on paper-based exercises given by the teacher. Exercises mainly included questions in the symbolic world. However, some of them can be viewed as questions in the embodied world through the correspondence between $\boldsymbol{R}^2$ and a plane or between $\boldsymbol{R}^3$ and the three-dimensional Euclidean space: determining linear (in)dependence of vectors in $\boldsymbol{R}^n$ ($n = 2, 3, 4$) or in polynomial spaces, determining whether a given set of vectors in $\boldsymbol{R}^n$ ($n = 2, 3, 4$) or in polynomial spaces is a basis, finding a basis and the dimension of given subspaces in $\boldsymbol{R}^n$ ($n = 2, 3, 4$) or in polynomial spaces, etc. Many questions were computational. Some of them were related to the geometric instruction given in the lecture part, and they can be answered with geometrical reasoning.

4.B. Design of tasks

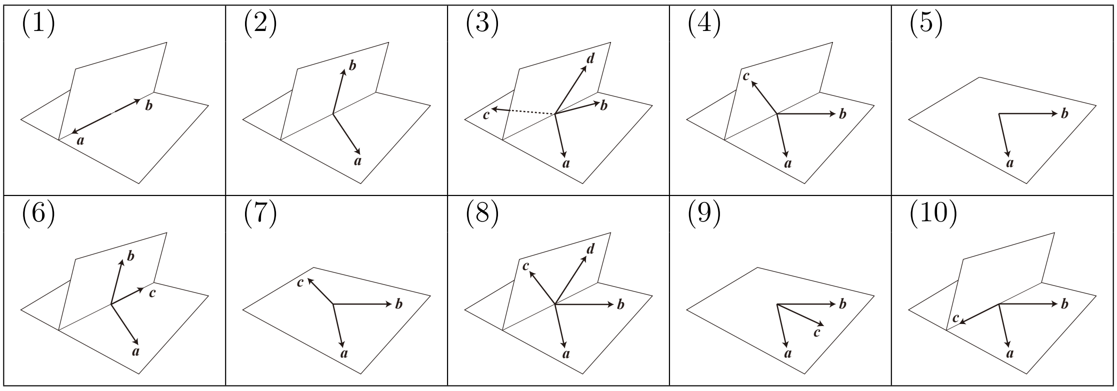
The following four tasks, which were translated from Japanese, were used in this study. They were designed to investigate students' understanding of dimension, linear (in)dependence, and basis. In order to address the first research question **RQ1**, we designed two types of tasks: tasks in the embodied world and tasks in the symbolic world. Task 1, Task 2, and Task 4 Q1 are tasks in the embodied world. Task 3 and Task 4 Q2 are tasks in the symbolic world. In order to address the second research question **RQ2**, the same items were used in Task 2 and Task 4; hence, correct answers for Task 2 were not shown to the students during the lessons.

Task 1. Answer the following questions. If you do not know (or if you have not learned), write your answer as "I don't know."

(1) What examples of one dimension, two dimensions, and three dimensions come to your mind? Describe the image you have for each of them using figures and words freely.

(2) For vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n, \boldsymbol{v}_{n+1}$, assume that vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ span an n-dimensional space V, and that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n, \boldsymbol{v}_{n+1}$ span an $(n+1)$-dimensional space W. When you draw a picture showing this situation, what kind of picture do you draw? Draw a picture of your image.

Task 2. Determine whether spatial vectors given in each picture are linearly independent. Note that each vector lies on a line or a plane shown in the picture. (If there are multiple planes, each vector lies on one of them.)



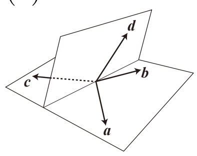
Task 3. (Q1) For vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ in a vector space V over K, describe two conditions (in the definition of basis) for $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ to be a basis of V. Write your answer in the answer columns (A) and (B).
(Q2) Determine whether the following set of vectors is a basis. If it is not a basis, answer which condition that you described in Q1 is not satisfied. In the latter case, write your answer by using a symbol A or B, and write "A, B" in both cases. (Vector spaces are as follows: (1) $\boldsymbol{R}^4$, (2) $\boldsymbol{R}^3$, (3) $\boldsymbol{R}^2$, (4) $\boldsymbol{R}^3$, (5) the space of polynomials $f(x)$ with degree less than 3 whose coefficients are in $\boldsymbol{R}$, (6) the space of polynomials $f(x)$ with degree less than 2 whose coefficients are in $\boldsymbol{R}$.)

(1) $\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ (2) $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}$ (3) $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}$

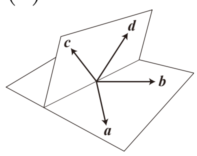
(4) $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\2\end{pmatrix}, \begin{pmatrix}-1\\-1\\-5\end{pmatrix}$ (5) $x+1, x^2$ (6) $x-1, x+1$

Task 4. (Q1) Determine whether spatial vectors given in each picture are linearly independent and describe the reason.

(1)



(2)



(Q2) Determine whether the given vectors in $\boldsymbol{R}^3$ are linearly independent and describe the reason.

(1) $\boldsymbol{a}=\begin{pmatrix}1\\1\\0\end{pmatrix}, \boldsymbol{b}=\begin{pmatrix}1\\2\\0\end{pmatrix}, \boldsymbol{c}=\begin{pmatrix}0\\1\\1\end{pmatrix}, \boldsymbol{d}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ (2) $\boldsymbol{a}=\begin{pmatrix}1\\1\\0\end{pmatrix}, \boldsymbol{b}=\begin{pmatrix}1\\2\\0\end{pmatrix}, \boldsymbol{c}=\begin{pmatrix}-2\\-3\\0\end{pmatrix}, \boldsymbol{d}=\begin{pmatrix}0\\0\\1\end{pmatrix}$

4.C. A priori analysis of tasks

Task 1 and Task 2 are pretests conducted at the beginning of the semester. Task 1(1) can be answered as "line," "plane," and "space." Task 1(2) is a non-routine task to examine whether students have an understanding that V is contained in W, or W extends outside of V as a space. Task 2 includes all of the important cases of at most four spatial vectors regarding linear (in)dependence. Items in Task 2 are the same ones that we used in our previous study (Kawazoe & Okamoto, 2016). According to the previous results, Task 2 (8) was expected to be difficult for the participants. Task 2 (8) contains four vectors, and any three of them do not lie on the same plane; hence, it cannot be reduced to the case of at most three vectors. Task 2 (3) also contains four vectors, but it can be reduced to the case of three vectors because vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ lie on the same plane. The terms "dimension," "span," and "linearly independent" were used in the texts in these tasks. As the participants were in their second year or higher, they had already learned these terms in their first year.

The aim of Task 3 is to investigate students' understanding of the definition of basis. Sets of vectors listed in (1)–(6) were all taken from the symbolic world. For any of these sets of vectors, one can determine their linear (in)dependence without computation. Only (2) and (6) are bases, and the others are not.

In Task 4, Q1 is a task in the embodied world, and Q2 is a task in the symbolic world. The two pictures in Q1 were taken from Task 2. According to the result of our previous study (Kawazoe & Okamoto, 2016), determining linear (in)dependence of four spatial vectors is problematic. Q1(1) and Q2(2) present essentially the same situation, and Q1(2) and Q2(1) likewise. Q1(1) and Q1(2) can be answered by drawing vectors representing linear combinations, or by using the fact on vector subspaces spanned by two or three vectors. Q2(1) and Q2(2) can be answered by using numerical computation (with or without the use of the Gaussian elimination), but they also can be answered with geometrical reasoning, such as, "four vectors cannot be linear independent in $\boldsymbol{R}^3$," or "three vectors on the same plane are linearly dependent."

4.D. Methodology and data collection

We implemented four-week lessons whose design is described in Section 4.A. Task 1 and Task 2 were offered at the beginning of the first lesson. Task 3 was at the third week, and Task 4 was at the beginning of the fifth week. Participants' answers for Task 1 were qualitatively analyzed whether they understood the dimension at most three and the increment of dimension. Participants' reasoning for Q1 and Q2 of Task 4 were qualitatively analyzed based on the classification of reasonings of linear (in)dependence given in Section 2.A. For other tasks, participants' answers were evaluated depending on their correctness. Subsequently, the relations between the results of these tasks were investigated by using a statistical analysis in order to address the research questions. Regarding the first research question **RQ1**, the relations between the results of tasks in the embodied world and the ones in the symbolic world were analyzed. Regarding the second research question **RQ2**, the relation between the result of Task 2 and that of Task 4 Q1 was analyzed.

The study was conducted in the fall semester in the 2018 academic year. All data were collected during the first five weeks in the linear algebra class for engineering students who had failed in the previous year or before. The number of students in the class was 58. Among the 58 students, 38 of them completed all the tasks, from Task 1 to Task 4. In this study, the data of the 38 participants was qualitatively and statistically analyzed.

5. Results

First, we present the results of each task and distinguish several groups of students depending on the results. Second, we statistically analyze the relations between the results of each task by using the groups determined in the first analysis.

5.A. Results of each task and students' groups

5.A.a. Results of Task 1

For Task 1 (1), almost all participants described their understanding of dimension 1, 2, 3, as a "line," "plane," and "space," respectively. For Task 1 (2), only 11 (28.9%) could draw their image of increment of dimension as extending outside the space. We set two groups according to the result of Task 1 (2): $GI+$ is the group of 11 participants who provided a geometric image of increment of dimension (Figure 3), $GI-$ is the group of the others.

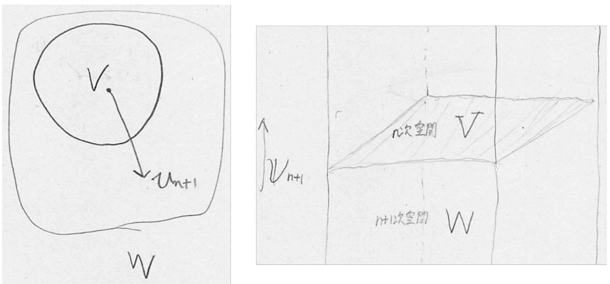


Figure 3: Examples of students' geometric images of increment of dimension

5.A.b. Results of Task 2

The percentages of correct answers for Task 2 were as follows: (1) 97.4%, (2) 94.7%, (3) 65.8%, (4) 97.4%, (5) 89.5%, (6) 94.7%, (7) 86.8%, (8) 52.6%, (9) 89.5%, (10) 86.8%. Therefore, the percentages of correctness for (3) and (8) were significantly lower than for the others. The pictures of (3) and (8) contain four vectors. The number of vectors in the others is less than four. The result of Task 2 was almost the same as the one in our previous study (Kawazoe & Okamoto, 2016), except for the result of (3). In the previous study, the percentage of correct answers for (3) was 84.5%. The reason for this difference is not clear, but it may have some relation to the fact that all participant in the present study have failed to pass the course in their first year. The median of the number of correct answers per participant was 9. We set two groups according to the result of Task 2: $GV+$ is the group of participants who answered correctly to more than 8 questions, and $GV-$ is the group of the others.

5.A.c. Results of Task 3

For Q1, the number of participants who could describe two conditions in the definition of basis correctly was 23 (60.5%). Although 34 (89.5%) of the participants described linear independence of the vectors correctly as one of the conditions, 24 (63.2%) of them described correctly that the vectors span V or that any vector in V can be represented as a linear combination of the vectors. 8 (21.1%) of them described $\dim V = n$ as one of the conditions, which is a wrong answer because $\dim V$ is defined after the definition of basis is introduced.

For Q2, although the percentages of correct answers for (2), (3), and (4) were high, those of (1), (5), and (6) were relatively low: (1) 78.9%, (2) 97.4%, (3) 94.7%, (4) 94.7%, (5) 78.9%, (6) 65.8%. This shows that understanding in the case of $\boldsymbol{R}^4$ and the space of polynomials were insufficient compared to the case of $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$. In this task, students were asked reasons only in the case when the given vectors are not basis, that is the case of (1), (3), (4), and (5). As for reasoning in these cases, we evaluated whether a participant could answer correctly based on the necessary and sufficient conditions to be a basis. Hence, for a participant who described $\dim V = n$ in Q1, we evaluated whether his/her answer for Q2 was logically correct based on his/her answer in Q1. The percentages of correct answers for reasoning were as follows: (1) 65.8%, (3) 63.2%, (4) 36.8%, (5) 65.8%. The median of the number of errors in Q2 (including errors in reasoning in the case of non-basis) per participant was 2. We set two groups according to the number of incorrect answers for Task 3 Q2: $B+$ is the group of participants whose incorrect answers were at most 2, and $B-$ is the group of the others.

5.A.d. Results of Task 4

The percentages of correct answers for Task 4 were as follows: Q1(1) 89.5%, Q1(2) 55.3%, Q2(1) 86.8%, Q2(2) 89.5%. The pictures in Q1(1) and Q1(2) are the same as in Task 2 (3) and Task 2 (8), respectively. Although the percentage of correct answers for Q1(2) remained low, the one for Q1(1) was much improved from the result of Task 2 (3). Though Q1(2) is essentially the same as Q2(1) from a geometrical viewpoint, their results were different.

With reference to the levels of understanding of linear independence distinguished in Section 2.A, we identified the following types of reasoning of the students.

The types of reasoning in Q1 were as follows:

(e1) drawing an arrow obtained as a linear combination of some vectors;

(e2) discussing the existence of a linear combination to represent some vector without drawing, but not referring to a plane nor a space: e.g., "$\boldsymbol{d}$ is represented as a linear combination of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$;"

(e3) using a plane or a space correctly in the reasoning: e.g., "Since the three-dimensional space is represented by three linearly independent vectors, one of the vectors is not needed"; "Four vectors in the same three-dimensional space are linearly dependent"; "$\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ lie on the same plane; since a plane is two-dimensional, it is represented by two vectors."

The following were typical examples of wrong reasoning in Q1 that we observed:

(w1) "Any of the vectors cannot be represented as a linear combination of the others."

(w2) "Any two of the vectors are not parallel, so they are linearly independent."

(w3) "Any three of the vectors do not lie on the same plane, so they are linearly independent."

(w4) "The plane which $\boldsymbol{a}$ and $\boldsymbol{b}$ lie on is different from the plane which $\boldsymbol{c}$ and $\boldsymbol{d}$ lie on, so they are linearly independent."

Among the above types of errors, (w2)-(w4) are conceptual errors; (w1) is a cognitive error.

The types of reasoning in Q2 were as follows:

(s1) finding coefficients of linear combination representing one of the vectors heuristically: e.g., "$\boldsymbol{d} = \boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c}$;"

(s2) using the Gaussian elimination;

(s3) same as (e3): e.g., "$\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ lie on the same plane"; "Four vectors in $\boldsymbol{R}^3$ are linearly dependent."

Regarding the reasoning in Q1 and Q2, we distinguished (e3) and (s3) from the others. The reason is as follows. Participants who used (e3) can be considered to have a strongly formed mental image of a geometric object obtained as a spanned space of spatial vectors, while participants who used (e2) can be considered to have formed a mental image of linear combination in the embodied world, but the image is not strong enough to immediately connect a set of all linear combinations to a geometric object. Participants who used (e1) can be considered not to have a sufficient mental image of linear combination. Since Q2 is a typical problem of linear independence of numerical vectors, (s1) and (s2) can be considered as *standard* reasonings for it. On the other hand, since Q2 is a problem in $\boldsymbol{R}^3$, it can be viewed as a problem of spatial vectors with the same geometric image as Q1. The reasoning of type (s3) can be considered as a geometric reasoning. As for participants using (s3), it can be considered that the symbolic world and the embodied world are strongly connected in their minds with respect to three-dimensional numerical vectors. From this viewpoint, we set the following groups according to the type of reasoning in each problem.

According to the reasoning in Q1, we set the following groups: For $j = 1, 2$, O_j+ is the group of participants using the reasoning of type (e3) for Q1(j), O_j- is the group of the others. According to the reasoning in Q2(j), we set the following groups: For $j = 1, 2$, GR_j+ is the group of participants using the reasoning of type (s3) for Q2(j), GR_j- is the group of the others.

5.B. Relations between the results of each task

By using the group setting defined in Section 5.A, we conducted statistical analyses to investigate relations between the results of each task. In the following statistical analyses, we used Fisher's exact test instead of the Chi-square test because of small numbers in cross-tabulation. Analyses showed in Section 5.B.a and 5.B.b address to the first research question **RQ1** and analyses showed in Section 5.B.c address to the second research question **RQ2**.

5.B.a. Relation between understandings in the embodied world and understanding of basis

Fisher's exact test indicated that having a geometric image of increment of dimension (Task 1 (2)) and the result of Task 3 Q2 were positively associated ($p < 0.05$, Table 2). Moreover, the use of a reasoning of type (e3) for Task 4 Q1(2) and the result of Task 3 Q2 were positively associated ($p < 0.05$, Table 3). However, we could not find any significant relation between $O_1+/-$ and $B+/-$.

	$B+$	$B-$
$GI+$	9	2
$GI-$	11	16

Table 2: Relation between the results of Task 1(2) and Task 3 Q2

	$B+$	$B-$
O_2+	8	1
O_2-	12	17

Table 3: Relation between the use of a reasoning of type (e3) for Task 4 Q1(2) and the result of Task 3 Q2

The analysis of Table 2 indicated that having a geometric image of increment of dimension was positively associated with understanding the basis in the symbolic world. The analysis of Table 3 indicated that having a strongly formed geometric image of spanned space of spatial vectors, especially for the case of four spatial vectors such that any three of them do not lie on the same plane, was positively associated with understanding the basis in the symbolic world.

5.B.b. Relation between understandings of linear (in)dependence in the embodied world and in the symbolic world

Fisher's exact test indicated that the use of a reasoning of type (e3) for Task 4 Q1(2) and the number of correct answers in determining linear (in)dependence in Task 4 were positively associated ($p < 0.01$, Table 4), where NC means the number of correct answers in determining linear (in)dependence in Task 4. However, we could not find any significant relation between $O_1+/-$ and the result of Task 4. Fisher's exact test also indicated that the use of geometrical reasoning (s3) for Task 4 Q2 and the number of correct answers in determining linear (in)dependence in Task 4 were positively associated ($p < 0.05$, Table 5), where $GR+ = GR_1 + \cup GR_2+$, $GR- = GR_1 - \cap GR_2-$, and NC is the same as in Table 4. Moreover, it indicated significant correlations for $GR_1+/-$ ($p < 0.05$) and for $GR_2+/-$ ($p < 0.05$).

	$NC = 4$	$NC < 4$
O_2+	8	1
O_2-	9	20

Table 4: Relation between the use of a reasoning of type (e3) for Task 4 Q1(2) and the result of Task 4

	$NC = 4$	$NC < 4$
$GR+$	10	5
$GR-$	7	16

Table 5: Relation between the use of geometrical reasoning (s3) for Task 4 Q2 and the result of Task 4

The analysis of Table 4 indicated that having a strongly formed geometric image of spanned space of spatial vectors, especially for the case of four spatial vectors such that any three of them do not lie on the same plane, was positively associated with understanding linear (in)dependence in both embodied and symbolic worlds. The analysis of Table 5 indicated that the use of geometrical reasoning in the symbolic world was positively associated with understanding linear (in)dependence in both embodied and symbolic worlds.

5.B.c. Difference of understanding of linear (in)dependence between before and after of four-week lessons

The images in Task 4 Q1(1) and Q1(2) are the same as the ones in Task 2 (3) and (8), respectively. McNemar's test indicated a significant difference between the results of Task 2 (3) and Task 4 Q1(1) ($p < 0.05$, Table 6), where the participants were divided into two groups depending on whether their answers for Task 2(3) were correct ($T_{2(3)}+$) or not ($T_{2(3)}-$), and they were divided into two groups depending on whether their answers for Task 4 Q1(1) were correct ($T_{4Q1(1)}+$) or not ($T_{4Q1(1)}-$). However, Fisher's exact test indicated that the result of Task 2 and the number of correct answers in determining linear (in)dependence in Task 4 Q1 were positively associated ($p < 0.01$, Table 7), where NC_{Q1} means the number of correct answers in determining linear (in)dependence in Task 4 Q1.

	$T_{4Q1(1)}+$	$T_{4Q1(1)}-$
$T_{2(3)}+$	23	2
$T_{2(3)}-$	11	2

Table 6: Relation between the results of Task 2 (3) and Task 4 Q1(1)

	$NC_{Q1} = 2$	$NC_{Q1} < 2$
$GV+$	16	7
$GV-$	3	12

Table 7: Relation between the results of Task 2 and Task 4 Q1

The analysis of Table 6 indicated that the understanding of linear dependence of four spatial vectors in Task 2 (3) had been improved during the four-week lessons. However, the result of Task 4 and the analysis of Table 7 indicated that the understanding of linear dependence of four spatial vectors in Task 2 (8) had not been improved.

6. Discussion

In this section, we discuss our research questions, based on the results from the previous section. First, we recall our research questions stated in Section 1:

> **RQ1**. Is geometrical understanding of linear (in)dependence in the embodied world related to understanding of linear (in)dependence and basis in the symbolic world?
>
> **RQ2**. Can geometrical understanding of linear (in)dependence in the embodied world, including the case of four spatial vectors, be improved by an instruction emphasizing a geometric image of linear (in)dependence?

As for the first research question **RQ1**, we observed some relations between understanding in the embodied world and understanding in the symbolic world. The results shown in Section 5.B.a and 5.B.b indicate positive correlations between understanding in the embodied world and understanding in the symbolic world. Note that these results only show correlations, which does not imply causal relationship. Hence, we should be careful in describing the effect of using geometric images for teaching linear algebra. The analysis of Table 2 might seem to support that students having a geometric image have some advantage in learning a linear algebra concept, because it shows a significant positive correlation between the result of pretest and the understanding of the notion of basis taught after the pretest. However, we should remind the reader that it was not the first time for the participants of this study to learn the notion of basis because they were students who failed to pass the course in their first year. Hence, it remains uncertain whether the understanding in the embodied world accelerates the understanding in the symbolic world or whether both types of understanding simultaneously develop by affecting each other. Thus, for the first research question, this study shows that geometrical understanding of linear (in)dependence in the embodied world is positively related to the understanding of linear (in)dependence and basis in the symbolic world, but we need more studies to investigate how the understanding in the embodied world affects to the learning of linear algebra concepts.

As for the second research question **RQ2**, we observed a limited effectiveness of the instruction that emphasized geometric images. As shown in Section 5.B.c, the understanding of linear dependence of four spatial vectors in Task 2 (3) had been improved during the four-week lessons. However, the understanding of linear dependence of four spatial vectors in Task 2 (8) had not been improved. The latter was surprising and also disappointing. Improving students' understanding of Task 2 (8) was more important because, as shown in Table 3 and Table 4, conceptual understanding of linear dependence in the case of Task 2 (8) was related to the understanding of basis and linear independence in the symbolic world. These results can be interpreted considering the following two possibilities: one is that the geometrical instruction implemented in this study was insufficient and can be improved; the other is that there is a limitation of students' perception even in the embodied world and it is cognitively hard to overcome such limitation. In the latter case, we should consider such limitation in teaching linear algebra, and it may lead us to reconsider how to design a linear algebra course using Tall's model of three worlds, especially to reconsider the balance and integration between geometric and algebraic presentation. However, the two possibilities need to be carefully examined in the future study.

Next, we determine whether the results on this study can provide any recommendations regarding the use of geometric images in teaching linear algebra. From the above discussion, we can say that it is difficult for students to view a space spanned by three linearly independent spatial vectors as the whole space. On the contrary, it is rather easy for students to view a spanned space that is a line or a plane as an object in the whole space. Hence, when we want to use geometric images of spanned space in a linear algebra class, we cannot assume that students would have a solid geometric understanding of a space spanned by three linearly independent spatial vectors as the whole space. This suggests that we should restrict using a spanned space that becomes a line or a plane when we use a geometric image of spanned space to help students learn concepts of linear algebra. We should remember that when we use a geometric image of a space spanned by three linearly independent spatial vectors, such space itself becomes a learning object for students.

Finally, we should mention the limitations of the study. First, the sample size was small. Second, the participants were not ordinary because they failed to pass the subject in the earlier years. Hence, further studies with a larger number of first-year students are needed.

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