ÉpiDEMES – 2023, 2 submitted 04/09/2021, accepted 27/09/2022, revised 08/05/2023

What is the place of examples in the teaching of abstract algebra?

Julie Stalder

Abstract. Teaching abstract algebra, seen as the study of structures and properties of structures, at a university level appears to be a challenge for both students and faculty. Some professors describe this passage through abstraction for the student as a “killing game”, an “impassable wall” or a “leap through abstraction”. In this paper, based on a case study (Candy, 2020) we investigate the choices of professors teaching abstract algebra at university. These professors were chosen because they teach abstract algebra at the university in France, and they gave us access to their course corpus and agreed to be interviewed. In this article, we choose to study in particular the teaching of the concept of ideal. We rely on an epistemological analysis to highlight its central role in the construction of abstract algebra. Then, using the Anthropological Theory of the Didactic (Chevallard, 1999), we will try to specify the place of examples and exercises in the students’ praxis.

Keywords. Abstract Algebra, Anthropological Theory of the Didactics, Ideal of a ring, Epistemology, Structuralist praxeologies.

Résumé. Enseigner l’algèbre abstraite, vue comme l’étude des structures et des propriétés des structures, à un niveau universitaire apparaît être un challenge pour les corps étudiant et professoral. Les professeurs de notre étude décrivent ce passage à travers l’abstraction pour le corps étudiant comme un « jeu de massacre », un « mur infranchissable » ou encore « un saut à travers l’abstraction ». Dans cet article, en appui sur une étude de cas (Candy, 2020) nous allons étudier les choix de professeurs enseignant l’algèbre abstraite à l’université. Ces professeurs ont été choisis car ils enseignent l’algèbre abstraite à l’université en France, de plus ils nous ont donné accès à leur corpus de cours et ont accepté d’être interviewés. Dans cet article, nous choisirons d’étudier en particulier l’enseignement du concept d’idéal. Une analyse épistémologique nous permettra de mettre en évidence son rôle central dans la construction de l’algèbre abstraite. Puis, en utilisant la Théorie Anthropologique du Didactique (Chevallard, 1999), nous chercherons à préciser la place des exemples et exercices dans la praxis du corps étudiant.

Mots-clés. Algèbre abstraite, Théorie Anthropologique du Didactique, Idéal d’un anneau, Épistémologie, Praxéologies structuralistes.

Contents

1. Introduction

Our study focuses on the teaching of abstract algebra, seen as the study of structures and properties of structures, at the university. To obtain results on the place of examples in the teaching of abstract algebra at a fine granularity level, we have chosen to focus our research on the teaching of the concept of ideal. The epistemological analysis which follows this introduction will aim at putting forward the relevance of this choice. Before that, it appears important in this introduction to highlight the previous works on the teaching of algebraic concepts at the university.

The notion of *Formalising, Unifying and Generalising* (FUG) concepts is introduced by Robinet (1984). Robert (1987, p. 3) points out that, in higher education, "the concepts to be acquired are generalising, unifying and formalising concepts: they have been developed by mathematicians after many particular problems have been solved (in a particular way) and often correspond to a new formalisation". The notion of *Simplifier* in the term FUGS appeared a little later with the work of Dorier on linear algebra (Dorier, 1990). Following a historical analysis of the emergence of the concepts of linear algebra, Dorier concludes that

> The use of the axiomatic theory of vector spaces as a privileged (or even unique) framework for the study of concepts and results in linear algebra is a recent discovery and choice. This choice essentially obeys a concern for organisation, unification and simplification of the study of different fields with similar tools and methods. (Dorier, 1990, p. 89)

Robert (1986) mentions the fact that not all concepts have the same status, and these differences certainly entail differences in acquisition. She thus specifies that "linear algebra, convergences, R, have a formalising, unifying and generalising status (c.f. Robinet); in fact, historically, these notions have been introduced, it seems to us, as an outcome, whereas the solutions of part of the problems concerned were already working well in each particular case" (Robert, 1986, p. 9, our translation). She then points to the difficulty of finding open-ended problems that would allow students to mobilise the concept as a tool before the course.

Robert and Robinet, in a study of linear algebra (Robert & Robinet, 1987, our translation), point out that "for a majority of students, finally, linear algebra is nothing but a catalogue of very abstruse notions that they are unable to imagine; moreover, they are submerged under an avalanche of new words, new symbols, new definitions and new theorems”. Thus, according to these authors, linear algebra concepts that are FUGS concepts "drown" students. They find themselves trying to make sense of these concepts whose formalisation is already difficult for them. The above quote suggests that students only see the axiomatized side of these concepts and cannot immediately access the simplification, unification and generalization that these concepts provide. This observation by Robert and Robinet is, in our opinion, valid for abstract algebra, whose formalisation characteristics are the same as for linear algebra.

For Robert (1987), the FUGS status of a concept is a characteristic of concepts in post-compulsory education, which implies that at the time of introduction of the concept, it is very difficult to find problems, accessible to students, where the concepts intervene in an optimal way. She, therefore, proposes that, in order to teach these concepts, one should give, after the course,

> problems that allow students to work on several frameworks in which their knowledge is unequal. We also retain the idea of the importance of reinvestment, including in problems apparently relating to another field, allowing the acquisition of tools that are available and not only mobilisable. (Robert, 1987, p. 12, our translation)

In (Jovignot, 2017) we had shown that the concept of ideal belongs to this class of FUGS knowledge. This research on FUGS concepts and the teaching of linear algebra highlights the complexity of finding fundamental situations to introduce these concepts due to their epistemological nature.

Regarding abstract algebra, Simpson and Stehliková (2006) explain that

> the power of much abstract algebra comes from this shift - working from the generality of the definitions to discover further relationships which tell us about properties which may have been hidden in examples of the structures. (Simpson & Stehlíková, 2006, p. 351)

In their article, they focus on the shift of attention from the familiar and specific of objects and operations to the interrelationships between objects expressed by the properties of operations (Simpson & Stehlíková, 2006, p. 352). This is what they call ‘apprehending the structure’.

Durand-Guerrier, Hausberger and Spitalas (2015) provide some insights into this structure. The article presents the analysis of a questionnaire proposed to third year undergraduate students at the University of Montpellier 2. After an epistemological analysis, they define the notion of paradigmatic example: "an example (or a class of examples) is paradigmatic when it allows us to identify a set of concepts and operative processes that have a general scope and can be implemented on other examples according to the same pattern" (Durand-Guerrier et al., 2015, p. 7, our translation).

To illustrate this notion, they cite $\mathbb{Z}$ or $K[X]$ as paradigmatic examples of Euclidean rings or $\mathbb{Z}[i]$ as paradigmatic for the extension of the arithmetic of $\mathbb{Z}$ and the construction of an abstract theory of ring divisibility. According to the authors:

> with regard to the learning of algebraic structures, the setting up of these paradigmatic examples is all the more fundamental as the conceptual mode of thinking that is

> mobilised by abstract algebra constitutes a paradigm shift relative to previous algebraic practices (historically, the algebra of equations and in the school context, those of high school) (Durand-Guerrier et al., 2015, p. 7, our translation)

The results of their study include the following (Durand-Guerrier et al., 2015, pp. 42-43): the articulation between abstract formal definitions and concrete object domains is difficult for students. They are, for example, not able to produce complex examples if they have not worked on them recently. The authors conclude that “the epistemological analysis highlights the fundamental role of the articulation between axiomatic definitions and examples”. (Durand-Guerrier et al., 2015, p. 44, our translation).

Fukawa-Connelly and Newton (2014) investigate the place of examples in the teaching of abstract algebra. Their paper begins with a review of the literature that justifies the importance of illustration through examples. Based on the study of a lecture-based introduction to the basic concepts of abstract algebra, they highlight, among others, the following results: first, the teacher uses several group examples by varying the different group classes (e.g. commutative or non-commutative, finite or infinite groups, or counterexamples of groups). Thus, according to Fukawa-Connelly and Newton (2014, p. 342): “the students had the opportunity to develop a well-defined example neighbourhood as they were exposed to a collection of examples in which many aspects of the examples were allowed to vary”. They also show that in his teaching, the teacher under study uses examples in different functions: firstly to illustrate definitions and also to instantiate propositions after introducing them. He [the teacher] also

> […] introduced the idea via a concrete example by asking, “What properties of the set and operation are necessary in order to be able to solve this equation?” Finally, Dr. P used examples to motivate claims (he was not observed generalizing from examples, so we have chosen “motivate” rather than “create”) before giving their formal statement, such as when he used solving equations in the integers to motivate the question of whether cancellation is possible in groups (Fukawa-Connelly & Newton, 2014, p. 344).

This research thus illustrates the importance of examples in the teaching of abstract algebra. However, they are limited to examples given by the teacher. In our study, we aim, in a complementary way, to study the use of examples in the problems the students have to solve (type of tasks and praxis) and not only in the explanation given by lecturers (logos).

The research discussed in this section has pointed at the fact that finding problems accessible to students that make sense of the concept of ideal appears complex, due to its FUGS nature. However, the works of Durand-Guerrier et al. (2015) and Fukawa-Connelly and Newton (2014) highlight the role of examples in the learning of abstract algebra. This leads to our research question in this article: what is the place of examples, in the activities proposed to students for the teaching of abstract algebra in Bachelor’s degrees in France?

To answer this question we will start with an epistemological analysis of the concept of ideal, which will allow us to situate the role of examples in the birth of this concept and more generally of abstract algebra. Then, we will present the theoretical framework, and in particular the elements specific to abstract algebra. Based on this framework, we will present, in the following part, the methodology of the study as well as the analyses of the corpus. Finally, in the conclusion, we will provide some answers to our research question.

2. Epistemological analysis

In this part, we are be able to retrace a complete epistemological analysis (that the reader can find in (Candy, 2020)) but we will cite the main milestones of the birth of the concept of ideal and more widely of the abstract algebra. We chose to use Corry (2004) as main resource because, in his book, he highlighted the role of the concept of ideal. However, the reader must be aware that this choice implies a particular point of view on the construction of the abstract algebra and thus does not aim at a faithful historical reconstruction but allows us to throw light on the construction of the concept of ideal in interrelation with other concepts of the structuralist algebra. In particular, we will link the formal character of the concepts and the place of examples in the construction of abstract algebra. This is why we will only use the study of the body of knowledge[1] (Corry, 2004) and not the analyse of the image of knowledge (Corry, 2004). This epistemological analysis intends to allow us, as mathematicians or mathematics education specialists, to get rid of the "illusion of transparency" (Artigue, 1991) which surrounds the concepts of modern algebra and the concept of ideal in particular, which our contemporary use tends to naturalize. This illusion of transparency, in the case of ideals, leads teachers to think that the concept, as they currently use it, has always been used and formalised as such.

2.A. Formal character

We will see in the following sections that the teachers in this study sometimes go so far as to teach the formal character of concepts first, as enabling the capacity for abstraction. This part of the epistemological analysis wishes to highlight that having the formal definition alone does not appear to be sufficient to motivate the use of the structure in the mathematical uses. Corry (2004, p. 32) explains, "the fact that many abstract definitions of mathematical concepts had already been formulated by that time seems not to have in itself sufficed to re-orient research in the abstract direction". He adds that the use of axiomatic allows for the effective treatment of structuralist theories, but that the developments of these theories

> […] were not directly connected with the use of the modern axiomatic method or of abstract formulations in general, but rather they were dictated by the solutions proposed for the existing problems of the discipline. It was the fact that similar sorts of problems arose in the different disciplines now included under the heading of modern algebra, and that at the same time they were addressed from an increasingly common perspective, that led to the unified exposition of van der Waerden. The full adoption of the abstract axiomatic formulation was only a part of a wider development. (Corry, 2004, p. 52)

---

[1] According to Corry "the body of knowledge includes theories, 'facts', methods, open problems. The images of knowledge serve as guiding principles, or selectors. […] Thus the images of knowledge cover both cognitive and normative views of scientists concerning their own discipline" (Corry, 2004, pp. 3-4)

In order to better apprehend what guides the structures, we will concentrate the epistemological analysis on the development of the concept of ideal. This will allow us, however, to obtain results that can be applied more widely to abstract algebra because, as Corry explains about the case of ideals

From the beginning of its development, its interplay with other, newly introduced algebraic concepts, such as fields, modules, groups, lattices, and polynomial rings, was clearly manifest. In fact, this interplay was central to the main concerns of the theory. Thus the theory of ideals had a rich pre-history, characterized by strong interconnections with other incipient structures, even before turning into a useful tool of central importance for the theory of abstract rings. (Corry, 2004, p. 15)

2.B. The birth of the concept of ideal and abstract algebra

According to Corry (2004, p. 85), it was in 1844 that Kummer became interested in the set (now) called the cyclotomic field. He discovered that factoring the number $a_0 + a_1\theta + \cdots + a_n\,\theta^n$ where $a_i \in \mathbb{Z}$ and θ is a 23$^{\text{th}}$ root of the unity does not have a unique factorisation into prime factors. In order to overcome this uniqueness, he developed the theory of ideal numbers.

Inspired by Kummer's work, Dedekind, considering $\mathbb{Z}[\sqrt{-5}]$ will extract number systems whose properties are the following:

> "- The sums and differences of any two numbers of system m will always be numbers of the same system m.
>
> - Any product of a number of system m and a number of system o is a number of system m. [...]
>
> We shall now call any system m, composed of numbers of the domain o and enjoying the two properties I and II, an ideal." (Dedekind, 1877, pp. 84-85, our translation)

Thus, Dedekind gives a definition of the concept of ideal which is almost the modern definition we use except that it has been formalised by relying on the properties of $\mathbb{Z}[\sqrt{-5}]$. In the rest of the paper he will extend the definition of the concept to bodies of numbers. Thus, this extension allows, for example, the simplification of the proof of the quadratic reciprocity law.

Dedekind wrote four versions of his theory of ideals. According to Corry (2004), Dedekind's formulation becomes more and more structuralist. Indeed, his first editions have theorems stated in terms of elements of the ideals whereas from the second version the treatment is done in terms of ideals and no longer in terms of elements that compose them. Finally, in the 4$^{\text{th}}$ version Dedekind will add a part on the Galois theory. Thus, in his research, Dedekind works more at the level of sets of numbers than of numbers themselves. However, he takes support on the properties of particular sets. Major modifications will still be necessary to arrive at the concept of ideal as a structure as we know it today.

Hilbert's axiomatic programme will be part of these changes. In 1893, Hilbert published Grundlagen der Geometrie (Hilbert, 1899, cited by Corry, 2004) which had a huge impact on the image of mathematics and made his name associated with the modern axiomatic method. He presents his axioms for geometry formulated for three systems of indefinite objects (points, lines and planes) and the axioms establish the mutual relations that should be satisfied by these objects.

Moreover, the impact of Hilbert’s axiomatic research coupled with the “formalism” associated with his name in the framework of the so-called “foundational crisis” of the 1930s have been occasionally seen as promoting the view of mathematics as an empty, formal game. But as a matter of fact, Hilbert’s own axiomatic research was never guided by such a view, and he often opposed it explicitly. Hilbert’s own conception of axiomatics did not convey or encourage the formulation of arbitrary, abstract axiomatic systems as starting points for mathematical research. Rather, his work was directly motivated by the need to better define and understand existing mathematical and scientific theories. In Hilbert’s view, the definition of systems of abstract axioms and their analysis following the above described guidelines were meant to be conducted for established and elaborated mathematical entities. (Corry, 2004, p. 161)

It was not Hilbert who changed the image of algebra, although he apparently had the cards in his hand to do so. Two subsequent pieces of research played a major role in the construction of the image of algebra as a study of structures. These were Steinitz’s work on the theory of abstract fields and Fraenkel's work on the theory of abstract rings. Indeed, according to Corry (2004), Steinitz was the first to make an abstract investigation of the concept of field, guided by Hensel and Fraenkel’s p-adic numbers theory. He was also the first to make systematic use of set theory in his research on algebra. Thus, we witness a change of point of view: it is no longer necessary to start from number systems to construct these fields as was the case with Dedekind. This time, on the contrary, number systems appear as a specification of these fields. As for Fraenkel, the axiomatic treatment of g-adic number systems (where g is not prime) led him to the definition of an abstract ring in 1914 and the study of its properties. However, Fraenkel did not make the link with Dedekind’s work and in particular, the concepts of ideals or modules, which are, however, in modern algebra, related concepts.

It was in 1905 that Lasker was inspired by Dedekind’s work on ideals to work in the factorisation of polynomials. He introduces the notion of a prime ideal but, again, without pointing out the connection with a notion introduced by Dedekind. This notion explained that, in some cases, these ideals might not be powers of prime ideals contrary to what happens in the algebraic numbers. Corry (2004) points out that this proves that at that time the concept of ideal, for example, did not yet play a unifying role. The theory of polynomials was still independent of the theory of algebraic numbers.

Macaulay, in 1913, published results in Mathematische Annalen which refined those of Lasker by proving the uniqueness of the decomposition of polynomial ideals into primary ideals. Thus, the decomposition theorem of polynomial ideals into prime ideals is known as the Lasker-Macaulay theorem. However, as we have just seen, those studies still contextualised the concept of ideal and not investigated it in a more general way.

It was not until Emmy Noether, whose work on ideals began according to Corry (2004) in 1917. She brought the concept of the ideal into the modern form we use. In 1920, Noether published his first research results with ideals. Indeed, Noether and Schmeidler proved that in differential operators the decomposition theorem can be generalised by formulating it in terms of the least common divisor of ideals. In this paper, they establish the existence and uniqueness conditions of the decomposition into ideals.

It was in 1921 that Noether gave birth to abstract algebra as we know it. It will unify and generalise theories such as those of algebraic numbers or polynomials, thanks to the abstract concept of the ring. Noether (1921) begins by writing:

> the aim of this article is to transfer the theorems of decomposition of the relative integers or that of decomposition into ideals in the algebraic numbers into ideals in the commutative rings and even in the general rings. (Noether, 1921, p. 25, our translation)

Noether defines an ideal in an axiomatic way similar to the one we know today (Noether, 1921, p. 29). A footnote explains that she adopts Fraenkel's definition except that she deletes properties that she says are too restrictive. In this paper (i.e. Noether, 1921), she will focus mainly on the study of integral domain whose ideals are all generated by a finite number of elements. She then proves the equivalence between the finiteness of ideals and the ascending chain condition. Based on this condition, Noether proves several theorems of decomposition of ideals (for more details, see (Noether, 1921) or (Corry, 2004)) and states a theorem (Noether, 1921, p. 50) which, according to Corry, is the major innovation of this paper:

> Noether's result is therefore not only a generalized formulation of the known theorem on polynomials, but in fact a new result which could not have been attained in the particular case of polynomial rings. (Corry, 2004, p. 233).

Noether, in this paper, will then reformulate from an abstract point of view properties obtained in her paper with Schmeidler using the concept of module on a ring. She shows that an ideal is a special case of a module and then reformulates the concepts introduced on ideals in terms of modules.

In 1927, Noether (1927) gave a general formulation of the first and second isomorphism theorems for modules over a general ring. Thus, rings are no longer only interesting to study for the factorization properties they allow to be stated. They become a sector that raises structuralist questions.

According to Mc Larty :

> Noether entirely reconceived the scope of the theorems and the relation between them. Dedekind proved the homomorphism theorem for finite groups, and the isomorphism theorems for infinite additive subgroups of the complex numbers. Noether gave a uniform method of proving isomorphism theorems from homomorphism theorems for many categories of structures — all groups, commutative groups, groups or commutative groups with a given domain of operators, all rings, commutative rings, rings with operators, and more. (Mclarty, 2006, p. 226)

It is for this work that Noether is considered the mother of structuralist algebra and the creator of the concept of ideal as it is used today.

2.C. Epistemological conclusions

This analysis already allows us to get rid of the illusion of transparency that surrounds the concept of ideal. As in Jovignot (2017), the concept of ideal appears to be a FUGS concept. Moreover, the role of the formal character is well highlighted by the example of the creation of the concept of ideal and of its transformation to arrive at its modern definition. Indeed, we notice that the structuralist treatment as we know it today was not immediately accepted by the mathematic community. In addition, it is noteworthy to underline that the changes in the body of knowledge and the image of mathematics concerning structures have been long in coming. Thus, even if today this work in terms of formal axiomatic and structure has become a familiar way of thinking for us, this was not always the case, even for expert algebraists. A shift of point of view was necessary to allow for these modifications, and this was done by using examples (number fields, polynomials, differential operators). Moreover, as Lipschitz suggests to Dedekind for his theory of ideals, the presentation of the problems that the theory of ideals allows to solve could facilitate the acceptance of ideals theory. Thus, this theory, even for expert mathematicians of the time, is costly and to be accepted must be able to prove its power by showing its usefulness in many problems.

3. Theoretical framework

Our main theoretical framework is the anthropological theory of the didactic (ATD, Chevallard, 1999).

3.A. Praxeology

ATD postulates that all human activities and thus mathematical activities in particular, can be analysed in terms of praxeologies. A praxeology is a quadruplet $(T, \tau, \theta, \Theta)$ where T is a type of tasks to be performed; τ the technique to do it; θ the technology, i.e. a discourse that justifies the technique and Θ theory, i.e. a discourse that justifies the technology. The block (T, τ) is the bloc of the *praxis* and (θ, Θ) the block of the *logos*.

To illustrate the concept of praxeologies we will analyse the following exercise extracted from our corpus.

> **Exercise**. Let $(A, +, \times)$ be a ring. Let I, J be two ideals of A.
>
> Show that $I \cap J$ is an ideal of A.
>
> Show that $I + J$ is an ideal of A.

In this exercise, the type of tasks T is “show that a subset S of a ring A is an ideal”. The technique τ is “show that S satisfies the properties of the definition”. The technology θ includes the“definition of an ideal” and the theory Θ the generalities over the ring and the fields that supports the definition of an ideal at stake. For more information on the role of theories in abstract algebra, we refer the reader to Candy (2020).

3.B. Structuralist praxeology and transition in abstract algebra

The epistemological analysis of the construction of abstract algebra, but also didactic analyses lead Hausberger (2018) to put forward transitions that would be at stake in the construction of abstract algebra. The author (Hausberger, 2016, p. 302) introduces the notion of structuralist praxeology. It is a praxeology used in abstract algebra. It aims at the achievement of a type of tasks by placing itself at a level of generality. In doing so, it provides simplification, based on concepts and structuralist technological tools (e.g. properties or theorems about structure). In (Hausberger, 2018), the author gives two steps for the development of structuralist praxeologies. For example, the type of tasks T: “show that a ring is a principal ideal domain”, can first be handled by elementary techniques and the associated technology will be reduced to the definition of the structures. However, in examining the technique it will become apparent that Euclidean division is the key to the demonstration. Thus, we can replace the technique τ by a technique τ_s: “show the existence of a Euclid algorithm”. Thus, the logos block $[\theta;\Theta]$ incorporates a theorem that deals with the structure “any Euclidean ring is a principal ideal domain”. According to the author, this is a first transition from a praxeology $\Pi = (T;\tau)$ to a structuralist praxeology $[\Pi_S;\theta;\Theta] = [T;\ \tau_s;\theta;\Theta]$. Furthermore, the author highlights a second type of transition that leads to praxeologies whose praxis and logos lie in the abstract. Following our example, students might encounter tasks like “show that a Noetherian integral domain such that any maximal ideal is principal is a PID”. Hausberger (2018) specifies the transitions of (Winsløw, 2008) in the framework of structuralist algebra: the praxis block of praxeology Π' is built on the block of logos of earlier structuralist praxeologies $[\Pi_{si},\theta_i;\Theta_i]$. In addition, Π' does not show any more concrete objects. These transitions will be illustrated in more detail in the rest of the article via praxeological analyses.

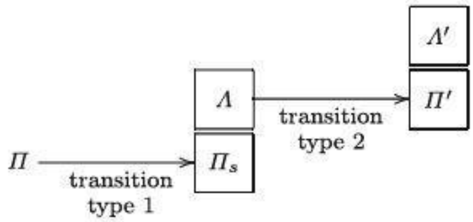

Figure 1 – The two types of transition in the Abstract Algebra curriculum at the university for a student who is learning structuralist praxeologies (Hausberger, 2018, p. 89)

Hausberger (2018) concludes that structuralist praxeologies take their place during the type 1 transition. As we will see, this is this transition that is at stake in our study.

4. The use of examples and the place of formal aspects in our corpus

4.A. Methodology

In Candy (2020), we studied the data of 9 French-speaking professors who teach an abstract algebra course dealing with ideals at the Bachelor's level. Among these 9 professors, 2 teach in the “classe préparatoire aux grandes écoles” mathematics and physic in France (i.e. during the second year of a curriculum that aims at allowing the integration of students into French engineering schools). We will call this institution CPGE. We will call the teachers P1 and P2. Two others are professors in the second year of the Bachelor’s degree in France, we will call them P3 and P5. Finally, three are professors in the third year of a Bachelor’s degree in France, we will call them P4, P6 and P7. Finally, two are professors in the first or second year of a Bachelor’s degree in Switzerland. In this article, we will only analyse the data from the French teachers with the idea of studying transitions.

We carried out detailed praxeological analyses of the course materials provided by the teachers. For the most part, we had access to the lectures, tutorials and their answer keys. To carry out our praxeological analysis, we studied each exercise that deals with the concept of ideal, as well as the technological elements that accompany them. For this, we used the ATD and, more specifically, the praxeological analysis. In addition, in order to strengthen this praxeological analysis, we conducted interviews with the teachers participating in the study. These interviews were analysed using a mixed method (Mayring, 2014). We previously fixed some coding thanks to the theoretical framework as the interviews were analysed. This analysis aimed to obtain various elements to specify the teachers' relationship to the teaching of structures or the use of examples in their lessons.

In the following, we present the results of the cross-analysis of the praxeological data and the interviews from the point of view of the management of transitions, the place of the formal and the role of examples in the teaching of the concept of ideal and more widely of abstract algebra.

4.B. Classes Préparatoires aux Grandes Écoles (CPGE)

The analyses of the official program of the mathematics-physics preparatory class show that structures are a learning objective. Indeed, we find a sector "usual algebraic structures" in which the first mention of the concept of ideal is made: "ideal of a commutative ring" in which we find the subject "divisibility relation in an integral domain". Later on, we find mention of the ideals of $\mathbb{Z}$ and $K[X]$ and in the sector "decomposition of an endomorphism". Thus, the program suggests, *a priori*, a teaching strategy that starts with the general definition of the structure for its application in $\mathbb{Z}$ and $K[X]$ as a paradigmatic example of Euclidean ring or principal ideal domain.

In the corpus of teacher P1, out of 18 tasks that deal with ideals, seven are abstract, i.e. they work in a general structure with properties. The introduction of structures is first formal and abstract and is, only later, worked on in paradigmatic examples (here $\mathbb{Z}[i]$ and decimal numbers as Euclidean rings[2], $\mathbb{Z}$ and $K[X]$ as principal ideal domain). Professor P2, on the other hand, chooses to introduce ideals first in the area of reduction of endomorphisms in order to give them motivation. Out of 24 tasks that deal with the concept of ideal, only 5 are abstract. Furthermore, the teacher uses examples as support for the construction of structures. For example, the type of task "demonstrate relations between inclusions of principal ideals and divisibility relations between generating elements of these ideals" is first presented contextualized in $\mathbb{Z}$ and then decontextualized and demonstrated in a general ring A. P2 gives importance to contextualization with paradigmatic examples before a generalization or institutionalization. Indeed, this phenomenon reappears several times in the course, notably before the introduction of the Great Common Divisor (GCD). Here, the game with the "nature of the ring" leads the students, based on the demonstration carried out in the paradigmatic example of integral ring $\mathbb{Z}$, to generalize the result to integral rings and thus demonstrate a structuralist theorem. This is a different didactic paradigm from that of P1, which, on the contrary, uses examples to illustrate technological notions stated in a general way beforehand.

In order to favour transition elements of type 1, P1 uses examples to show that a Euclidean ring is principal. Even if he does not go as far as the formal statement of this property, he will put forward, in an answer key, the spring of the proof of this property. Indeed, he will write, "the same structure is found in all Euclidean rings”. He wants to accompany the type 1 transition in his students. He explains, “I try to do as much as possible demonstrations that show that nothing is specific to integers or polynomials”. However, this support on paradigmatic examples remains punctual and comes only after a formal introduction of the concepts.

Thus, although the official curriculum is common to both teachers, they make different didactic choices regarding the use of examples. P1 starts with an abstract and formal chapter and then contextualises later. P2, on the other hand, first deals with concepts in a contextualised way with paradigmatic examples and then allows their formal and general introduction. We could explain these differences between these teachers by their different relationships (following the sense of Chevallard, 2003) to algebra. Indeed, in his interview, P1 explains, “it is the discipline that makes people find maths abstract that they never understood etc. and not at all analysis or probabilities or geometry”. P2 explains that “if I were to write an algebra book I would start with the problems and I wouldn't start with the structures [... that's] why we do algebra”. P1 thus emphasizes the abstract character while P2 tends to emphasize the tool character of the concept of ideal. P2 will explain that he has taken

> a freedom, when I discussed it with certain colleagues, it didn't seem justified to them to introduce the ideal independently of a chapter of algebra [...] for the people of my generation there was still for some the notion of structures was very important. They had a spirit, for some, we'll say caricatured: mathematics had to be organised, sometimes the discussions were rather folkloric.

4.C. Second year of the Bachelor’s degree

The analyses of the official programmes of the two B2 teachers show that the programme is mainly formulated in terms of properties and theorems. For example, there is mention of “GCD and LCM” or “Bezout and Gauss theorem”. P3 program did not explicitly mention the concept of ideal. P4 program mentioned it but contextualised to polynomials: “ideals and GCD of polynomials”. In both programmes, structures do not appear as a learning objective. The type 1 transition does not appear as a learning objective in this second year of the Bachelor’s degree.

Moreover, the praxeological analysis shows that the concept of ideal intervenes in a few tasks and few are contextualized. One calls contextualized a task that works in a contextualized structure, such as $K[X]$ as a Euclidean ring and not a general structure with properties. The first teacher (P3) has only 8 tasks that deal with ideals and only two are contextualised in $K[X]$ (once as a paradigmatic example of a principal ideal domain and the other as a paradigmatic example of a Euclidean ring). These tasks consist mainly of a formal axiomatic game from the definitions, such as, for example, “showing that a set is an ideal”. Thus, in the framework of the concept of ideal, the praxeological baggage constructed is weak. It is based on very few tasks, a majority of which are decontextualized to allow a formal axiomatic game. This could be explained in part by P3’s relationship to structuralist algebra and its teaching. Indeed, the teacher explains that it is necessary for students “to know how to play with the definition”. He “thinks that abstract algebra also starts with things like that where we just have an ideal, we don’t really know, but it’s something that has properties and we play with it [...] yes, here we don’t say it’s such and such polynomial times such and such polynomial, it's the set of all the things like that". The second teacher, P5, only deals with the concept of ideal in the parts of the course and there is no task in the students' praxis.

Thus, despite the absence in the syllabus of any mention of structure and the desire to bring properties in paradigmatic examples (such as $K[X]$) to work, teachers have preferred to introduce a formal chapter on structures. On the one hand, we can explain it by the difficulty of using other paradigmatic examples. Indeed, P3 explained that he could not use $\mathbb{Z}$ as an example because $\mathbb{Z}$ is in the program of another module independent of his own. On the other hand, in their interviews, starting with a formal treatment of structures appears to be a way of facilitating the type 1 transition that takes place between the second and third year of the Bachelor's degree. They describe this transition as an "impassable wall" or "conceptual leap". To this end, they adopted the same didactic strategy: introduce structures in second year of bachelor, insisting on their formal character (they are "vocabulary" or "words"). Moreover, despite the programmes suggesting contextualised work, and in particular to the rings of polynomials, the few tasks present favoured decontextualized formal work. Thus, contrary to the CPGE teachers, in the second year of the Bachelor's degree, the work is essentially formal, almost detached from the examples, and this in a desire to encourage work in the abstraction to come in the third year of the Bachelor's degree.

4.D. Third year of the Bachelor's degree

P4 is the third year Bachelor teacher who teaches after P3. P4's syllabus includes both concepts and properties of concepts, but also specific examples (like $K[X]$ or $\mathbb{Z}[i]$). We could identify these examples with paradigmatic examples (of Euclidean ring or principal ideal domain) and it will be interesting to see if the teacher builds on these examples to generalise as P2 did. Secondly, this syllabus and contrary to those of the second year of the Bachelor, explicitly mentioned ideals many times. Structures are a learning objective of this course.

The praxeological analyses of P4 show that out of 45 tasks, 10 are abstract tasks. Within some of the praxeologies, we find occurrences of type of tasks first contextualised and then decontextualized. This is the case, for example, of the type of tasks "show that an ideal I of a ring A is prime". The first occurrences of this type of tasks are contextualised to particular ideals of $\mathbb{Z}[X]$ (exercise 7, below). Then follows the type of tasks that aims at proving prime avoidance lemmas based on the definition of a prime ideal and thus extending the definition of a prime ideal in terms of inclusions of ideals (exercise 8, below). It seems to us that this last praxeology leads to a type 2 transition. Indeed, the praxeology in Exercise 8 below is abstract and the praxis block of this praxeology builds on earlier contextualised structuralist praxeologies constructed in Exercise 7.

7**. Ideals of $\mathbb{Z}[X]$**.

1) Show that (2) is prime but not maximal.

2) Show that (X) is prime but not maximal.

3) Show that $(2, X)$ is not principal. Show that is a maximal ideal.

4) Show that (X^2+1) is prime but not maximal. Do you recognize $\mathbb{Z}[X]/(X^2+1)$?

8. **Prime ideals**. Let A a ring and P and ideal of A ($P \neq A$). Show that P is prime if and only if for all I, J ideals of A, we have: $IJ \subset P \Longrightarrow I \subset P$ or $J \subset P$.

Figure 2 – Illustration of tasks leading to type 2 transition

Finally, we find the paradigmatic examples of classical principal rings of $K[X]$ and $\mathbb{Z}$, as well as of non principal ideals ((X,Y) in $K[X,Y]$, $(2,X)$ in $\mathbb{Z}[X]$ in the exercise 7). These examples had been highlighted as paradigmatic by the teacher in the interview ("already, there are examples that are a bit 'pie in the sky', where you have to look at, the ideals of $\mathbb{Z}$ and $K[X]$. You have to look at $K[X,Y]$ or $\mathbb{Z}[X]$ to show that there are rings that are not principal ideal domains"). Thus, P4 uses paradigmatic examples to illustrate formal properties or punctually to lead to a type 2 transition.

P6 is the teacher who teaches after P5. P6's programme is formulated in terms of structures. There is no mention of examples except for "ring of polynomials and its universal property", where the domain of objects is considered with a view inherited from category theory (notion of "universal property"). Ideals, operations on ideals, principality properties and prime and maximal ideals come at the beginning of the course.

The interview of P6 shows that the teacher in charge of the seminars takes into account the ecology of the course to choose his exercises while being guided by the will to contextualise abstract notions "by putting concrete exercises and by making them work on familiar objects". Thus, in the praxeologies for which the student is responsible, only 9 out of 41 are abstract. One of the praxeologies, abstract by nature, is the one generated by the task type "demonstrate formal properties of operations on ideals". The task "determine the result of operations on principal ideals" is first worked on with a technique based on the dictionary, on particular cases. Then the abstract task which /will consist in the demonstration of the existence of the GCD, via ideals, in a commutative principal ring will allow the construction of a technology "in a principal ideal domain $(a) + (b) =(gcd\ (a,b))$" which makes it possible to calculate the sum of two ideals by using the GCD

One of the exercises of the seminars aims at proving that $\mathbb{Z}[i]$ is a principal ideal domain by proving first that it is Euclidean. The work is done on a paradigmatic example, opening the door to a transition of type 1 when, in the course, the demonstration is carried out in the general case. The success of the transition, on this example, thus rests on a good articulation between the lectures and the seminars, because two different professors give those without necessarily a strong coordination. Moreover, in the following of the seminars, the teacher in charge do not used the structuralist theorems for the demonstrations. On the contrary, in the sectors "maximal ideals" and "prime ideals" the technology is mainly the definition of these concepts, leading to a handwork on the objects. Thus, the type 1 transition within the P6 course seems unfinished at the end of the module and this is partly due to the coordination between lectures and tutorials.

P7 is a professor who gives the first course in which ideals take place. The syllabus of the P7 course is very succinct and consists of a list of mathematical concepts related to or being structures, e.g. "homomorphisms, ideals, subrings, polynomial ring, Euclidean rings". In the part "rings and fields", the concept of ideal is explicitly mentioned. There is only one contextualisation mentioned by the programme in this part and that is "polynomial ring". The programme as it stands therefore leaves a lot of freedom to the teacher but emphasises concepts and structures. The structures are therefore a learning objective of this programme.

In the P7 teaching material, out of the 19 tasks, only 5 are abstract, the others are contextualised. The type of tasks “tasks which mobilise the ideals only via the quotient $\mathbb{Z}/n\mathbb{Z}$ ” (8 occurrences on the corpus) will be generalised then to “tasks which mobilise the ideals only via a quotient A/I” by a play on the variables A and I. This is the only type of task for which we observe this phenomenon. For the others, the occurrences of the types of tasks are always 1 and even in the sector “first and maximal ideals”, the tasks are always contextualized to $\mathbb{R}[X, Y]$. The study of the contextualisation of the tasks highlights the weak vitality of the back and forth between work on structures and work on objects. While the course introduces concepts at a high level of generality in a structuralist way, in the exercises the work is mainly done at the structuralist level 1. Moreover, the dissemination of tasks between the types of tasks will lead to a difficulty in systematising the techniques. We linked this dissemination to one of his learning objectives with regard to ideals: P7 expects students to reach what he calls the “creative” level, in the sense that “the student appropriates the reasoning and is able to modify it to make another”.

Half of the structuralist praxeologies have a definition in their technology and require “hand-written” verifications. Moreover, even if the other praxeologies have structuralist theorems as technology, the unique occurrences of the tasks of these praxeologies make it difficult to understand the power of these theorems from a generalizing point of view. Thus, it seems to us that there are possible impediments to a type 1 transition. The interview with P7 suggests that the lack of time given the official programme could explain this phenomenon.

Thanks to the praxeological analysis and the interviews, we can conclude that in the second year of the Bachelor’s degree, structures appear as a learning objective in their own right for some teachers. The type 1 transition takes place between the first and second year of the Bachelor's degree. Instead of being formulated in terms of objects, the programmes are this time treated in terms of structure. All third-year Bachelor teachers in this corpus start by formally introducing structures and then using them in contextualised praxeologies. The interviews show that this is strongly related to the teachers’ institutional relationships to the teaching of structuralist algebra. Thus, the teachers only use paradigmatic examples to enable the creation of structures using the potential for generalisation.

5. Conclusion

Our analyses have highlighted that the type 1 transition appears to take place between the second year of the Bachelor’s degree (or CPGE) and the third year of the Bachelor’s degree. Occasionally, teachers use examples to facilitate transitions. For example, the lecturers work with the same praxeology on examples before generalising by putting forward what makes the spring of this praxeology as a generalising principle.

However, we note that due to many constraints, teachers can no longer rely on paradigmatic examples. Let us highlight some of these constraints as presented in (Candy, 2020). On the one hand, the separation of modules can sometimes make it difficult to use paradigmatic examples. Indeed, as students do not all follow the same modules, they may have different praxeological baggage in terms of paradigmatic examples. This coupled with other constraints such as time or the number of ECTS[2] credits complicates the support on paradigmatic examples for teachers.

Finally, we have seen that the teachers in the second year of the Bachelor's degree wanted to favour the type 1 transition by preparing the students for abstraction. To do this, they opt, under the effect of their reports on the teaching of algebra, for a formal presentation of structures and a predominantly axiomatic work. They reduce the use of structures to a 'vocabulary', which, according to these lecturers, will facilitate their use in the third year of the Bachelor's degree. However, Candy (2020) highlights the fact that third-year Bachelor's professors still feel that students have difficulty moving towards abstraction despite an abstract introduction. One of the CPGE lecturers who first teaches structures in contexts and then decontextualizes them, pointed out a strong cultural connection to the teaching of abstract algebra. The formal character of structures appears culturally as the necessary premise for abstraction. Yet both our epistemological analysis, previous work on examples (Durand-Guerrier et al., 2015) as well as theoretical framework (Hausberger, 2018) highlight the place of examples in the construction of abstraction. In order to foster teaching strategies that use paradigmatic examples as a catalyst for abstraction, it appears that we should modify the institutional and personal relationships to the teaching of algebra.

References

Artigue, M. (1991). Épistémologie et didactique. *Recherches en Didactique des Mathématiques, 10*(2.3), 241–286.

Candy, J. (2020). *Étude de la transposition didactique du concept d'idéal : écologie des savoirs et problématique de l'entrée dans la pensée structuraliste en France et en Suisse romande*. Thèse de Doctorat. Montpellier : Université de Montpellier.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*, *19*(2), 221-266.

Chevallard, Y. (2003). Approche anthropologique du rapport au savoir et didactique des mathématiques. In S. Maury & M. Caillot (Eds.) *Troisièmes journées d'étude franco-québecquoises* (pp. 81-104).

Corry, L. (2004). *Modern Algebra and the Rise of Mathematical Structures* (2nd éd.). Birkhäuser Basel. https://doi.org/10.1007/978-3-0348-7917-0

Dedekind, R. (1877). Sur la théorie des nombres entiers algébriques. *Bulletin des sciences mathématiques et astronomiques, 1*(1), 69–92.

Dorier, J.-L. (1990). Analyse historique de l'émergence des concepts élémentaires d'algèbre linéaire. *Cahier de DIDIREM 7*. http://docs.irem.univ-paris-diderot.fr/up/publications/IPS97070.pdf

Durand-Guerrier, V., Hausberger, T., Spitalas, C. (2015). Définitions et exemples : prérequis pour l'apprentissage de l'algèbre moderne. *Annales de Didactiques et de Sciences Cognitives, 20,* 101-148.

[2] European Credit Transfer and Accumulation System

Fukawa-Connelly, T., & Newton, C. (2014). Analyzing the teaching of advanced mathematics courses via the enacted example space. *Educational Studies in Mathematics, 87*(3), 323-349.

Hausberger, T. (2016). A propos des praxéologies structuralistes en Algèbre Abstraite. In E. Nardi, C. Winsløw & T. Hausberger (Eds.) *First Congress of the International Network for Didactic Research in University Mathematics* (pp. 296–305). Montpellier. https://hal.archives-ouvertes.fr/hal-01322982/document

Hausberger, T. (2018). Structuralist Praxeologies as a Research Program on the Teaching and Learning of Abstract Algebra. *International Journal of Research in Undergraduate Mathematics Education, 4*(1), 74–93.

Jovignot, J. (2017) L'analyse épistémologique du concept d'idéal et ses apports à l'étude didactique. In M. Bächtold, V. Durand-Guerrier & V. Munier (Eds.) *Épistémologie & didactique. Synthèses et études de cas en mathématiques et en sciences expérimentales* (pp. 61-73). Presses Universitaires de Franche-Comlté.

Mayring, P. (2014). *Qualitative Content Analysis: theoretical foundation, basic procedures and software solution*. Klagenfurt. https://nbn-resolving.org/urn:nbn:de:0168-ssoar-395173

Mclarty, C. (2006). Emmy Noether's 'Set Theoric' Topology: From Dedekind to the rise of functors. In J. Gray & J. Ferreirós (Eds.) *The Architecture of Modern Mathematics: Essays in history and philosophy* (pp. 211–235). Oxford.

Noether, E. (1921). Idealtheorie in Ringbereichen. *Mathematische Annalen, 83*(1), 24–66. doi:10.1007/BF01464225

Noether, E. (1927). Abstrakter Aufbau der Idealtheorie in algebraischen Zahl- und Funktionenkörpern. *Mathematische Annalen, 96*, 26–61. https://gdz.sub.uni-goettingen.de/id/PPN235181684_0096?tify={%22pages%22:[30],%22view%22:%22export%22}

Robert, A. (1986). Didactique dans l'enseignement supérieur : une démarche. *Cahier de didactique des mathématiques, 28*. http://docs.irem.univ-paris-diderot.fr/up/CDM_28_Aline_Robert_-_Didactique_dans_lenseignement_sup%c3%a9rieur_-_une_d%c3%a9marche.pdf

Robert, A. (1987). De quelques spécificités de l'enseignement des mathématiques dans l'enseignement post-obligatoire. *Cahier de didactique des mathématiques, 47*. http://docs.irem.univ-paris-diderot.fr/up/publications/IPS00019.pdf

Robert, A., & Robinet, J. (1987). Quelques résultats sur l'enseignement de l'algèbre linéaire. *Cahier de didactique des mathématiques, 53*.

Robinet, J. (1984). *Ingénierie didactique de l'élémentaire au supérieur*. Thèse de Doctorat. Paris : Université Paris VII. https://hal.science/tel-01250673

Simpson, A., & Stehlíková, N. (2006). Apprehending Mathematical Structure: A Case Study of Coming to Understand a Commutative Ring. *Educational Studies in Mathematics, 61*(3), 347-371. http://link.springer.com/10.1007/s10649-006-1300-y

Winsløw, C. (2008). Transformer la théorie en tâches : la transition du concret à l'abstrait en analyse réelle. In A. Rouchier & I. Bloch (Eds.) *Actes de la XIIIème école d'été en didactique des mathématiques*. La Pensée Sauvage.

Affiliation

Julie Stalder
Haute-Ecole Pédagogique du Valais
e-mail: julie.stalder@edufr.ch