Mechanisms for mathematical modelling: the study and research paths at university level

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Abstract. This paper addresses the problem of integrating mathematical modelling into first-year mathematics courses at university level. Our research focuses on identifying *mechanisms* that facilitate the dissemination of mathematical modelling in university mathematics education. Within the framework of the anthropological theory of the didactic (ATD), our work over recent decades has focused on the design, implementation, and analysis of the *study and research paths* (SRP) as a teaching device persuading a double purpose: making students aware of the rationale of mathematical contents through the experience of modelling activities; and connecting these mathematical contents through a whole modelling process. We draw upon empirical findings from the implementation of an SRP on population dynamics with first-year students at university level, and its 'migration' to other university settings, to identify valuable mechanisms for integrating mathematical modelling into university institutions. More concretely, we analyse the mechanisms facilitating two central dialectics for the SRP and for modelling: the dialectics of questions and answers and that of media and *milieu*.

Keywords. Mathematical modelling, Anthropological theory of the didactic, Study and research paths, Mechanisms, Dialectics.

Résumé. Cet article aborde le problème de l'intégration de la modélisation mathématique dans les cours de mathématiques de première année au niveau universitaire. Notre recherche se concentre sur l'identification des *mécanismes* qui facilitent la diffusion de la modélisation mathématique dans l'enseignement des mathématiques à l'université. Dans le cadre de la théorie anthropologique du didactique (TAD), notre travail des dernières décennies s'est concentré sur la conception, la mise en œuvre et l'analyse des *parcours d'étude et de recherche* (PER) en tant que dispositif d'enseignement ayant un double objectif : faire prendre conscience aux étudiants de la logique des contenus mathématiques à travers l'expérience des activités de modélisation ; et relier ces contenus mathématiques à travers l'ensemble du processus de modélisation. Nous nous appuyons sur les résultats empiriques de la mise en œuvre d'un PER sur la dynamique des populations avec des étudiants de première année à l'université, et sa « migration » vers d'autres contextes universitaires, pour identifier des mécanismes qui paraissent pour la modélisation mathématique dans les institutions universitaires. En particulier, nous analysons les mécanismes facilitant deux dialectiques centrales pour le PER et la modélisation : la dialectique des questions et des réponses et celle des médias et du milieu.

Keywords. Modélisation mathématique, Théorie anthropologique du didactique, Parcours d'Études et de Recherche, Mécanismes, Dialectiques.

1. Introduction

The starting point of this research is the problem of inquiring into the conditions that can help that mathematical modelling can be integrated and developed in the teaching and learning of mathematics into current educational systems. Researchers and practitioners agreed on the necessity of proposing alternative teaching practices, particularly at the university level, where mathematics could be taught as a service subject becoming an essential modelling tool to enquire into the study of real problems (Holton, 2001; Niss, 2001; Blum, 2015).

This change requires moving from a more traditional pedagogical paradigm of transmission of knowledge, which mainly focuses on introducing students to already built mathematical knowledge, to a paradigm of inquiry where solving problematic questions leads to learning processes and motivates the study of new knowledge. This is what Y. Chevallard characterise as the paradigm of 'visiting works', or of 'visiting monuments', where students are invited to visit mathematical knowledge muted from its rationale, towards the paradigm of 'questioning the world' where questions are placed at the core of knowledge construction and mathematics understood as a modelling tool to investigate questions to collectively provide answers.

> "[In the paradigm of visiting monuments] Because these questions are usually hushed up—visiting a monument is no place to raise "What for?" or "So what?" questions—, students are reduced to almost mere spectators, even when educators passionately urge them to "enjoy" the pure spectacle of mathematical works" (Chevallard, 2015, p. 174)

In the research domain known as 'Applications and modelling', some advances have been made to show how modelling activities can be successfully performed, under certain suitable conditions in different educational levels and curricular frames (Blum, 2015; Burkhardt, 2006, 2018; Doerr & Lesh, 2011, among others). However, although school institutions and researchers agreed that modelling should play an important role towards a change of pedagogical paradigm, the real situation in schools and universities is not satisfactory (Stillman et al., 2013) and the dissemination and long-term survival of these teaching proposals based on modelling follows as a big challenge for mathematics education (Galbraith, 2007).

> "In the case of applications and modelling a shared excitement unites many who have been enthused about early experiences in the field, for example when students unleash latent power that for whatever reason had remained fettered in their previous mathematical life. However, this very exhilaration can work against further progress, both individually, and particularly at a system level, by creating a sense of adequate achievement that obscures the reality that there is so much more to do." (Op. Cit., p. 79).

In this paper, we focus on first-year university courses in mathematics training for non-specialists. As highlighted by González-Martín et al. (2021), the interaction between different specialisms becomes increasingly important in the research field of university teaching: mathematics and mathematical modelling for non-specialists. The aim is twofold: to make sense of observed teaching and learning, and to devise new and more relevant and efficient didactical approaches, for instance, to help students engage in mathematical modelling.

In this context, we analysed the most prevalent way of teaching mathematics for natural sciences at university level. As stated in Barquero et al. (2013), one of the main drawbacks underlined of such teaching organisation is that it regularly creates difficulties due to the disconnections between the taught contents and their poor motivation about their uses and functionalities with the rest of scientific disciplines. It is also highlighted that even though mathematical models appear in the syllabi of most of the courses, teaching mathematical models often comes at the end of the process if there is time left for it. Then, the prevalent model at the university level is that modelling represents in most cases a mere 'application' of some pre-established knowledge, leaving little room for the process of questioning, building, and validating mathematical models. This prevalent epistemological model was called and characterized by Barquero et al. (2013) as 'applicationism'.

Our research, developed in the framework of the anthropological theory of the didactic (ATD henceforth), faces the question of how to change the institutional relation of university teaching about modelling by looking into which *mechanisms* could help modelling to have an explicit and central role in the teaching of mathematics. More concretely, the research questions here addressed can be formulated in the following terms:

> RQ1: How can we design teaching devices where mathematical modelling has an explicit and crucial role, emerging from generating questions and linking mathematical contents that appear as models to inquire into these questions?
>
> RQ2: Which *mechanisms* can help modelling play this role in mathematics teaching at university?

To address our research questions, we focus on the proposal of the *study and research paths* (SRP) that have appeared in the framework of the ATD, as teaching devices to move towards the paradigm of questioning the world (Chevallard, 2015; Bosch, 2018). In the next section, we present some of the main traits of the SRP, their design principles, and the conceptualisation of mathematical modelling within this framework. Our object of analysis is the design and implementation of modelling practices through the proposal of the SRP. In sections 3-5, we consider a case study with an SRP about population dynamics, which was implemented for 5 consecutive years with first-year university students. This case study is one of the first SRP implemented at university level. The mathematical and didactic designs from this initial SRP were later adapted to be able to migrate to other university settings, starting from other initial questions and introducing some variations on their implementation modalities. We thus analyse the successive changes on the SRP to be adapted in three university contexts, helping us to investigate the *mechanism* that can favour that modelling could be effectively integrated and developed at university level.

2. The study and research paths and the role of mathematical modelling

Chevallard (2006 and 2015) introduced the proposal of the *study and research paths* (SRP) as a general model for designing and analysing teaching and learning processes in a change of pedagogical paradigm, from a 'monumentalist' approach to a paradigm of 'questioning the world'. Previous investigations have shown how modelling can be integrated at different school levels through the SRP proposal (Bosch, 2018; Jessen, 2017; Florensa et al., 2018). In the framework of the ATD, most of the mathematical activity can be described in terms of modelling. In other words, in any mathematical activity, there can be raised some questions to be addressed, which leads to the construction of mathematical models to provide possible answers. This looking for answers requires going through a modelling process, which can be characterised by different stages (Chevallard, 1989), similarly as the description provided through the modelling cycles (e.g., Borromeo Ferri, 2006), without decomposing too much the stages it includes:

> 1. Delimitation of the system (mathematical or extra-mathematical) where the questions emerge and the selection of the variables to study; 2. Formulation of

hypothesis and construction of mathematical models; 3. Work with the mathematical models to provide answers; 4. Validation of models by contrasting them to the initial system.

In this process, numerous questions may arise, leading to new modelling processes. Following the research of García et al. (2006), modelling activity is proposed to be reformulated as a process of construction and articulation of mathematical organisations of increasing complexity in a progressive and recursive process. Within this study, the authors propose to characterise mathematical modelling as processes of reconstructions and connection of *praxeologies* of increasing complexity that should emerge from questioning the rationale of the praxeologies that are to be reconstructed or connected (op. cit., p. 243). Central to this modelling process is to investigate how to make systems and models emerge, evolve, transform, and connect, at each step, in more complex and complete structures that allow mathematical knowledge to appear functional (provide answers to something) and connected. We aim to analyse the proposal of the SRP and the role that modelling plays within it, focusing on two pivotal questions concerning two central dialectics:

How to enhance the dialectics between *posing questions* and *looking for answers* –the dialectics of *questions-answers*– as the driving force behind mathematical modelling? Which dialectics between *media-milieu* is necessary for students and lecturers to facilitate the appropriate development of mathematical modelling?

2.1. The generating question and the dialectics of *questions-answers*

One of the main traits of SRPs is that they start from a lively question of potential concern for the students and teachers called the *generating question*, expressed as Q_0. When the community of students and teacher(s) decides to pursue it, Q_0 evolves by opening many other derived questions, expressed as Q_i. The main purpose is that students, guided by the teacher, provide answer(s) to Q_0 together with the derived questions. The study of Q_i leads to looking for successive temporary answers A_i. Then, the structure of the SRP can be synthesized as a tree of linked questions and answers (Q_i, A_i), which traces the possible 'paths' to be followed in the effective experimentation of the SRP.

In the next sections, we use the questions-answers structure of the SRP to describe the *a priori* design and the *a posteriori* analysis of the SRP we focus on. Addressing Q_0 allows students to go through several modelling cycles in which they iteratively express-test-and-revise their answers and make them progress beyond first-iteration responses. In this process, the questions considered do not disappear, and neither the answers (hypothesis, models build, used, validated…), which take an integral part of the built knowledge.

This first layer of the design of the SRP allows us to deal with different aspects. On the one hand, to enquire into the potential of the generating questions Q_0 and to trace the possible path to be followed in the effective implementation of an SRP. That is, to foresee if Q_0 is 'fertile' enough and to sketch its 'life expectation' by describing the derived questions that can be opened and the extra- and intra-mathematical tools and models that the study of Q_0 and Q_i may ask for. On the other hand, to provide researchers and teachers with alternative epistemological models to describe the mathematical activity, which is usually described following the logic of the mathematics contents, to be now described in terms of the interplay between questions, mathematical models, and possible answers.

The necessity of considering these alternative epistemological models for mathematical (or modelling) activity remains in the necessity of overcoming some important constraints linked to the dominant 'museographic' paradigm of visiting works, absent of the possible questions that can be at the origin and *raison d'être* for its consideration. Moreover, as expressed in the works of Orange (2005, 2007), there exists an important constraint about the few possibilities of integrating a real 'problematising' activity (posing questions) in classrooms. This supports the first working assumption on the proposal of the SRP about the necessity of integrating rich *dialectics of questions-answers* or posing questions and discussing the answers as an engine for the SRP and rich and complete modelling activity. In section 6, we focus on some of the mechanisms that have helped with this first dialectics to be placed at the core of the SRP.

2.2. Progressive enrichment of the milieu and the dialectics of *media-milieus*

There are many possible ways to approach a generating question depending on the derived questions opened and the answers provided. But also, the SRP can significantly vary according to the different *means* that are made available during the teaching and learning process. Then, another dialectics, the one of *media-milieus* helps describe the dynamics of the SRP (Chevallard, 2011; Kidron, et al., 2014). The term *media* is used in the same sense as any source of information (one person, textbooks, papers, lectures, websites, class notes, etc.) used to obtain previously established knowledge or answers that students consider relevant to solve the questions raised during the process. They provide pieces of knowledge that cannot always be used directly but should be validated and adapted by checking them against a didactic *milieu* (in the sense of the Theory of Didactic Situations, TDS, Brousseau, 1997), that is, against the set of empirical objects and knowledge that is already available for the students and can act as a 'piece of nature' to them.

In front of the generating question, and to provide a collective answer $A^\heartsuit$ to it, X, the students, and Y the teachers must bring into a didactic *milieu* (M) composed made up of all the "tools" the use of which seems indispensable or at least useful (Chevallard, 2019). This milieu can be composed of derived questions (Q_i), answers that the class built (A_i) or pre-labelled answers that already exist outside in different media ($A_i^\Diamond$), mathematical work/objects (W_j) that help to evaluate the pertinence of certain answers, experimental data (D_l), among other possible elements. Chevallard (2011, 2019) synthesized the activity started in a didactic system $S(X; Y; Q_0)$, which is formed to study a certain question Q with a class X and a teacher Y who guides the study with the semi-developed form of the *Herbartian schema* that can be written as follows:

$$[S(X; Y; Q_0) \Rightarrow M\,] \hookrightarrow A^\heartsuit$$

where the didactic *milieu* M can be composed of the elements mentioned before. This schema can be further developed by replacing M with the following elements:

$$[S(X; Y; Q_0) \Rightarrow \{\, A_i^\Diamond, W_j\,, Q_n\,, A_n\,, D_p\,\}] \hookrightarrow A^\heartsuit$$

By using the Herbartian schema, one can analyse different teaching and learning practices, in particular, modelling teaching practices. For instance, Wozniak (2012) analyse several teaching practices with 'Giant shoe' modelling activity at Primary school level which shows how underdeveloped is the dialectics of *media-milieus*. In this activity, the teacher is the mean media, which provides students the questions and tools or models to be used (or to be applied), and where students' answers are reduced to the application of pre-given models decided and provided by the teacher.

> It therefore appears that there can be no construction of models in the sense of modelling process described by Chevallard, without the implementation of a *dialectics of media-milieus* that makes it possible to clarify the problematization. If the hypotheses on which the model is based are not stated, or if they are stated without being questioned or legitimized, without their validity being explored, then we will consider that the modelling process has not been fully developed and that the model construction building stage has been partially completed. (op. cit., p. 72, our translation)

On the contrary, if the teachers propose to students to inquire beyond the pure application of pre-established models, there can be questions and decisions about which hypotheses are better to consider, what models are proposed by experts in different media, how to validate and make models evolve, and so on. Then, the *dialectics of media-milieus* would be, at least, different, allowing a different degree of questioning and development of modelling for *X* and *Y*. In the last section, we focus on some of the mechanisms and levers that help to integrate this second essential dialectics at the core of the SRP.

2.3. New responsibilities and their distribution in the collective construction of $A^♥$

The SRP aims to promote the role of the class or study community (*X*, *Y*) where the group of students *X* may share a set of tasks, under the guidance of *Y* the teacher(s) and agree the responsibilities for each to assume. This displacement going from the individual to the community has many important consequences to make the existence of mathematical modelling possible. On the one hand, the collective study of questions provides the opportunity of defending responses produced by the community, instead of accepting the imposition of the official answers. On the other hand, this work requires a new distribution of responsibilities and, consequently, changes in the prevailing didactic and pedagogical contracts between the students and the teacher. The teacher thus must assume a new role of acting like the leader of the study process, instead of lecturing the students. The students might share with the teacher(s) responsibilities on raising and agreeing on questions to address; formulating hypotheses; searching and discussing different models; collecting and selecting data; choosing the relevant mathematical tools to validate and reformulate models; writing and defending reports with partial or final answers, and so on. It soon appears that the teaching culture at university does not offer a variety of teaching strategies for this purpose. The changes in the new distribution of responsibilities, and the need to establishing a new didactic contract for a collective approach to the inquiry and to the modelling activity is further explained through the analysis of the SRPs.

3. Research methodology and conditions for implementing the SRP

The research methodology followed for the design, implementation, and analysis of the SRP is the steps of the didactic engineering process (Artigue, 2014), which includes four main stages: the preliminary analysis, the *a priori*, *in vivo* and *a posteriori* analysis. As explained by Barquero and Bosch (2015), the starting point of the research is the integration of mathematical modelling in first-year university courses of Mathematics, in natural science university degrees in Barquero et al. (2013) or business administration, in Serrano et al. (2010, 2013).

When analysing what kind of mathematics is taught at this level, for the initial *preliminary analysis* of the existing conditions (and constraints), one could think (natural science, business administration) university degrees would offer favourable institutional conditions to teach mathematics as a modelling tool. However, this seems far away from reality: the dominant ideology is that modelling represents a mere application of some pre-established knowledge, leaving little room for the process of proposing, constructing, validating, and questioning mathematical models. The second stage is devoted to the *a priori* analysis of the teaching proposal, that is, to the mathematical and didactic design of the SRP. On the one hand, in the mathematical design, the SRP is described as a *map of questions and answers* that are derived from a generating question about population dynamics tracing the possible routes to be followed in the SRP implementation (Winsløw et al., 2013). In the following section, we summarize the *a priori* analysis of the SRP regarding population dynamics. The didactic design of the SRP appears to complement the previous description with a more precise organisation of each session: formulation of Q_i to address, set of data given, management of the possible responses and new questions, sharing of responsibilities between the students and instructors (as some the above mentioned), etc. The third stage of the process consists of the *in vivo* analysis with the implementation, observation, and data collection from the implemented SRP, which mainly came from students' team and individual reports, the teacher's written description of the work carried out during each session, worksheets given to the students, video recording of the workshop sessions, and brief questionnaire to the students at the end of the process. It constitutes the empirical base upon which rests the *a posteriori* analysis of the SRP, the fourth and last step, which consists of the validation and development of *a priori* design proposed by the two previous phases and reflects the role, participation, and dynamics generated by the students' groups and by the teacher.

In this paper, we focus on an SRP that was initially implemented for five academic years (from 2005/06 to 2009/10), with first-year students of technical engineering degree at the Universitat Autònoma de Barcelona. The implementation took place within the one-year 'Mathematical Foundations of Engineering' course, with about 40 students. Their designs were afterward adapted and migrated to other university institutions of Business Administration degrees (from 2006/07 to 2018/19). Table 1 summarizes the SRP topics, the subject, the university and university degrees, where they were implemented, the academic year, and some references for further information about these SRPs.

The common conditions of SRP1-SRP2 were that the SRPs ran in a teaching device called the 'mathematical modelling workshop'. The workshop ran over 2-hour weekly sessions during the whole course. In the case of SRP3, the modelling workshop was implemented in the transition from the first to the second term (1^{st} term: one-variable sequences, functions; 2^{nd} term: derivatives, integrals, and differential equations), aiming to show the use and applicability of some of the mathematical tools introduced in the first term or started introducing them for the second term. The instructors of the workshop were researchers in Didactics of Mathematics, also lecturers of the regular course. In the following section, we describe the initially designed SRP on population dynamics, which will be then used to highlight the adaptations of SRP2 and SRP3 into the other university contexts.

SRP	SRP topic	Subject, Level Degree, University	Period	References
SRP1	Population dynamics	*Mathematics*. 1st year Chemical engineering, UAB	2005-2010	Barquero et al. (2013, 2018)
SRP2	Users of Lunatic world networks	*Mathematics*. 1st year Business administration, IQS – URL	2006-2014	Serrano et al. (2010, 2013)
SRP3	Evolution of Facebook users	*Mathematics*. 1st year Business administration, Tecnocampus	2015-2019	Barquero et al. (2018)

Table 1 – List of experienced SRPs

4. The *a priori* analysis of the SRP on population dynamics

The starting point of the SRP is the study of the population evolution and how to predict its evolution. The initial generating question (Q_0) of the SRP can be formulated in the following terms: Given the size of a population over some periods, can we predict its size after n periods? Is it always possible to predict the long-term behaviour of the population size? What sort of assumptions about the population and its surroundings should be made? How can one create forecasts and test them?

In front of this question, one can initially assume that time t is measured in discrete units, and that the population size (x) at time t, x_t, depends, among other factors, on past states $x_{t-1}, x_{t-2}, \dots, x_{t-d}$ $(0 < d \leq t)$. Addressing Q_0 can lead to consider two main types of models: (1) when x_t only depends on x_{t-1}, that is a population with independent generations, where models based on recurrent sequences of order 1: $x_{t+1} = f(x_t)$ are considered, with f is a real-valued function of one variable, or (2) when x_t depends on the $d > 1$ past generation $x_{t-1}, x_{t-2}, \dots, x_{t-d}$ (population with mixed generations), one study recurrent sequences of order $d > 1$, which can be expressed as vector recurrent sequences with $X_{t+1} = f(X_t)$ where $X_0 = (x_0, x_1, \dots, x_{d-1})$ is the vector of the d initial generation sizes and $X_i = (x_{id}, x_{id+1}, \dots, x_{id+(d-1)})$ is the *i-th* vector of d generations where $0 \leq i \leq n$. If we assume that time t is measured as a continuous variable, we study the continuous evolution of the population, which has an analogous structure to the situations described above.

From these first considerations, two main branches on the design of the SRP are opened. On the one hand, the first branch focused on the *discrete models* where the models to be built are based on *recurrent sequences*, with order 1 or bigger than 1, depending on if we consider populations with independent generation or populations with mixed generation. The first case covers the sector of sequences and their convergence and the one of one-variable calculus; the second case covers the sector of linear algebra. On the other hand, a second branch focused on the *continuous models* where, depending on if we consider independent or mixed generation, it allows us to consideration of models based on ordinary differential equations (ODE) of order 1 or bigger than 1. In the following section, we include the description of two parts of both branches, combined with inputs from the real implementation of the SRP. One may note that the *a priori* design of the SRP, which will be described in terms of questions-answers, are dynamics structures, that is, with each implementation and its *a* *posteriori* analysis the initial design has been enriched and new questions, hypothesis, models, answers, etc. have been incorporated.

Before going into detail with the description of the modelling activity developed, its design shows different elements that collide with the dominant way of organising the teaching of mathematics at university level. First, the generating question is an open question without an immediate answer. It is expected to take a long time to be addressed, and it would imply changing the order of appearance of some contents of the regular course, following the logic of the questions opened in the workshop. Second, it was an open modelling activity where the groups of students could differ in their proposals. One of the central aims was to discuss and agree collectively the way to progress to elaborate a final $A^{\heartsuit}$.

5. Implementation of the SRP on population dynamics

The initial workshop session of the SRP started presenting some data corresponding to the evolution of the size of a population of pheasants in an isolated island, over 5 years (data extracted from Lack, 1967). The students were asked to analyse the data and provide an initial answer to the generating question Q_0. In the five SRP implementations, students took three main approaches to the problem: some groups tried finding the best polynomial interpolation to the data, other groups tried an exponential fit to the data and the rest tried using recurring sequences to model the population dynamics. In the class discussion, the three approaches were presented by the teams and the students, under the guidance of the instructor, decided that the discrete recurring sequences approach would be explored first, leaving the continuous approach for a later SRP (3rd branch of the SRP experienced during the 3rd term). This decision was taken mainly because when students were asked to explicit the different elements of the whole modelling praxeology: hypothesis assumed ($Q_{0.1}$), models built according to these hypotheses ($Q_{0.2}$), only the teams working on the discrete world and with models based on recurrent sequences could provide and justify $A_{0.1}$ and $A_{0.2}$ by formulating hypothesis over the absolute or relative rates of growth.

> $\boldsymbol{Q_{0.1}}$: What hypothesis about rates of growth can be formulated? According to them, which mathematical models can be considered to fit and forecast data? ($\boldsymbol{Q_{0.2}}$)

From here, during the following two months, we experienced the first branch of the SRP. It ran over eight weekly sessions of 2 hours each. The instructor with the students agreed to retake the proposals of using continuous models (mostly based on exponential functions), during the 2nd semester, after experiencing this 1st branch, and when the regular course started with the block of derivatives and differential calculus.

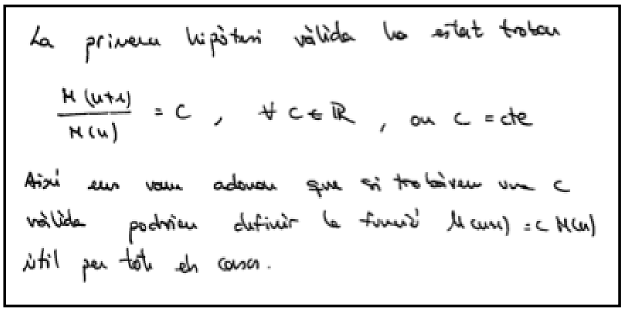
5.1. From the Malthusian model to the reformulation of the initial hypothesis

In the second workshop session, the class agreed on the notation to be used and the instructor introduced some requirements on the population to simplify the study, such as considering independent generations. x_n was defined as the size of the n-th generation of population X and the study of the population evolution was thus characterized by the study of the sequence $(x_n)_{n\in N}$. The assumption of independent generations led the class to consider several indicators of the population growth. There appeared the *relative rate of growth* of X between consecutive generations:

$r_n = \frac{(x_n - x_{n-1})}{x_{n-1}}$ or the net production rate: $i_{n+1} = \frac{x_{n+1}}{x_n}$. In all the implementations, some of the groups began by proposing one of the easiest assumptions about the growth of X (see Figure 1):

> ***H*1.1**: The rate of growth of the population is constant: $r_n = r,\ r \in \mathbb{R}$ and ***Q*1.1**: How does a population with a constant rate of growth evolve over time?

This assumption led to the construction of the first model (***M*1.1**) which is equivalent to the first-order recurrent equation: $x_{n+1} = r.x_n + x_n = (1+r).x_n = \alpha.x_n$, with $\alpha = 1 + r$. Given the initial population size $x_0 = c > 0$ and assuming that the relation is valid for any consecutive generation, the students could approach x_n using the equivalence: $x_{n+1} = \alpha.x_n \Leftrightarrow x_n = \alpha^n.c$. The exploratory study of M_1 allowed the students to provide a temporary answer A_1 to initial question Q_0 depending on the value of α: (1) If $\alpha < 1$ the population is wiped out; (2) if $\alpha = 1$, the population size remains constant independently of the time elapsed; and, finally, (3) if $\alpha > 1$, the population grows indefinitely (see Figure 2).

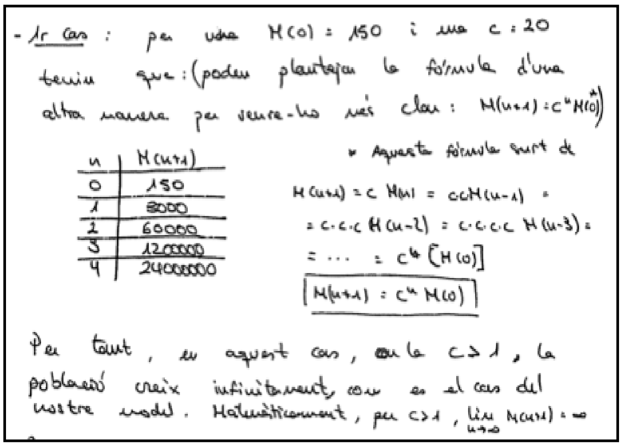


Translation: The first valid hypothesis has been to ask that

$$\frac{M(n+1)}{M(n)} = c, \forall c \in \mathbb{R}, \text{where } c = \text{constant}$$

We realized that, if we found a valid c, we could define the sequence $M(n+1) = c \cdot M(n)$ useful for any case.

Figure 1 – Formulation of H_1 about the relative rate of growth



Translation: 1st case: If we have $M(0) = 150$ and $c = 20$, we can use the previous formula in another way to get the result easier:

$M(n+1) = c^n \cdot M(0)$.

This formula comes from the following:

$$\begin{aligned} M(n+1) = c \cdot M(n) &= c \cdot c \cdot M(n-1) \\ &= c \cdot c \cdot c \cdot M(n-2) = \\ &= c \cdot c \cdot c \cdot c \cdot M(n-3) = \cdots \\ &= c^n \cdot M(0) \end{aligned}$$

Therefore, in this case, with $c > 1$, the population grows indefinitely, as our model shows.

Mathematically, if $c > 1$, $\lim_{n \to +\infty} M(n+1) = \infty$

Figure 2 – Numerical simulation of M_1 and conclusions about population dynamics

Students quickly realized that this first model (M_1) presented a clear limitation: the fact that the population grows indefinitely (case $\alpha > 1$) falls into the biological paradox of assuming the existence of infinite resources (Figure 2). In front of this, a new question was formulated as *how could we* *overcome this unrealistic fact?* At this point, most of the students concluded that it was more coherent to assume that r_n decreases depending on x_n, which was finally formalise as:

> ***H*1.2**: The size of the population cannot exceed a given maximum value K. Therefore, the rate of growth must decrease when the population size approaches this maximum value. For example, we can assume the simplest case of a rate of growth decreasing linearly with size. ***Q*1.2**: What are the dynamics of a population under $H_{1.2}$ conditions?

At some stages, it was necessary that the instructor of the workshop introduced new mathematical knowledge to help the students' progress in the study of $Q_{1.1}$ and $Q_{1.2}$. The first term of the regular course primarily focused on the study of elementary functions and their properties, so there were several important tools, such as the different kind of rates of growth, the definition of what a recurrent sequence is, how to numerically simulate sequences (through which experimental mean), etc. that were necessary to be formalized thanks to the work initiated throughout the workshop. Both instructors of the course met and agreed on the most optimal integration of the mathematical tools into the instructional process. They agreed on whether to incorporate them in the following workshop or lecture sessions. Furthermore, since the beginning, students were encouraged to seek sources for existing pre-stablished answers to the questions under examination. In several implementation, they brought forth information regarding the Malthusian or logistic models they had found in external resources. This practice enabled the class to assign names to these models and study the information contained into these external media.

5.2. From the logistic discrete model to a general functional model

One of the simplest models that satisfy ***H*1.2** is summarized by the recurrent equation $x_{n+1} = \alpha x_n \left(1 - \frac{x_n}{K}\right)$, known as the *logistic equation* (discrete) or *Verhulst model*. Once this second model ***M*1.2** was built, students tried to look for a closed-form equation, i.e., with a general formula $x_n = f(n)$ which, in contrast with the Malthusian model ($M_{1.1}$), the logistic equation does not admit. The numerical simulation, for some values of the coefficients α and K, constitutes the first *experimental mean* to simulate the population dynamics modelled by $M_{1.2}$.

This exploratory study provided the following experimental answer $A_{1.2}$: (1) If $\alpha < 1$, the population is wiped out; (2) if $1 < \alpha < 3$, the population size grows to an equilibrium situation; (3) if $\alpha > 3$, one finds cases which are difficult to analyse (for instance, there are cases where the population growth oscillates between several values (2, 4, 8, etc., or cases where the long-term behaviour is completely unpredictable). The institutionalization of these conjectures from different students' teams was pertinent to reveal the appearance of new derived questions, such as:

> ***Q*1.2.1**: What does the well-definition and convergence of x_n depend on, in the logistic model case? and ***Q*1.2.2**: What does the speed of convergence of x_n depend on?

At this point, it appeared necessary to consider a new model, including the previous ones and able to re-formulate and face the questions that had remained open (such as $Q_{1.2.1}$ and $Q_{1.2.2}$). The work performed so far can be described by ***Q*1.3** which leaded to the consideration of more general model based on a recurrent relationships of the form $x_{n+1} = f(x_n)$ where *f* represents the functional relationship between two consecutive generations of the population. If f is a linear or quadratic function we have the previously analysed models.

> ***Q*1.3**: What are the dynamics of the sequence x_n generated by the relationship $x_{n+1} = f(x_n)$ where f can be any C^1 function?

In the SRP implementation, after being introduced to the graphical techniques of sequence iteration (particularly, to the cobwebbing techniques, see Figure 3), the teams began by considering f as a linear function (the case of the Malthusian discrete model, $M_{1.1}$) to then face the case of f a quadratic function (the case of the logistic discrete model, $M_{1.2}$). Students used this new experimental *milieu* provided by the graphical simulation to validate the conjectures and answers they had suspected with the numerical simulation of these models.

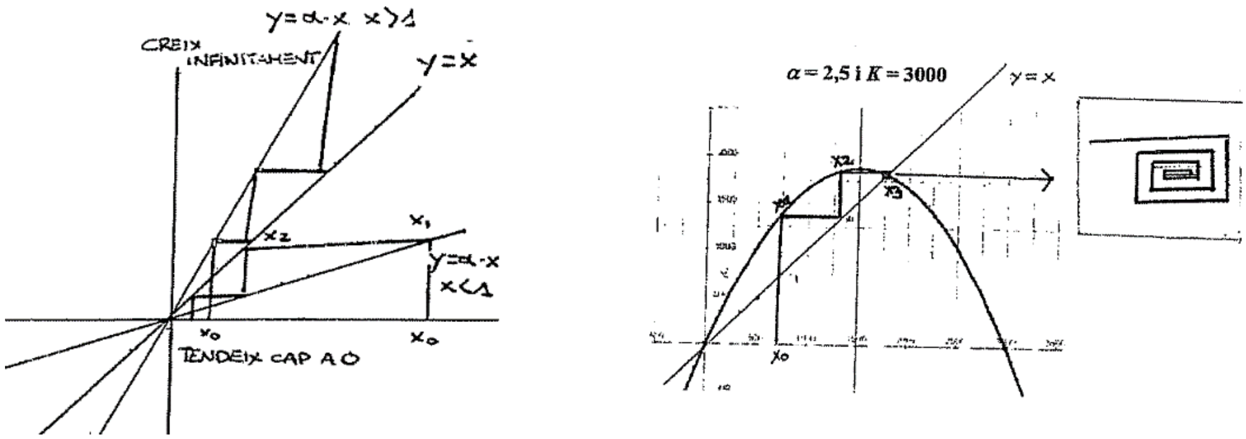


Figure 3 – Formulation of H1 about the relative rate of growth

During this phase, the analysis of the behaviour of $x_{n+1} = f(x_n)$ needed the use of more advanced mathematical tools, such as C^1-functions, graphical techniques of simulation, analysis of derivatives. Although these tools were not originally included in the course programme, the instructors opted to introduce them during the workshop and lectures' sessions to ensure that the students could understand and use then properly. This decision significantly enlarged the scope of the types of problems that could be addressed, including the expansion the kind of functions to consider. In all the five implementations conducted, time constraints prevented further exploration. However, on several occasions, teams proposed the consideration of other decreasing functions, which would entail the exploration of new models. For instance, the Ricker model by considering f an exponential decreasing function*;* or the Beverton-Holt model by considering f a rational function. While these proposals aimed to extend the initial design of the SRP, they remain unexperienced in practice.

5.3. From the discrete to the continuous modelling processes

As previously mentioned, it was in the second semester when the whole second branch of the SRP was developed, focused on the continuous models. This second part was focused on ***Q*2**, about which continuous models can be used to fit data and to provide forecasts about population dynamics. Two types of yeast populations were employed at this occasion: *Saccharomyces cerevisiae* and *Saccharomyces kéfir* populations (Carlson, 1913) living in independent containers.

During the second semester, students had overcome the initial resistance and progressively accepted a lot of new responsibilities they were asked to assume: defending their reports, posing new questions, looking for pre-existing answers outside the classroom frame, etc. While a detailed examination of the SRP development is beyond the scope of this discussion, some noteworthy aspects deserve emphasis. First, the parallel structure between the first and second branch of the SRP significantly enhanced students' autonomy in various steps of the modelling process. For instance, their ability to formulate hypotheses regarding the population growth, and to construct, simulate, and validate models was greatly facilitated by the previous work in the first SRP. Second, the parallel structure of both branches (see Figure 4), along with the students' fluency in the modelling process, led to the emergence of new and crucial questions, not foreseen in the initial design of the SRP. There appeared interesting questions about possible relationships between the hypothesis assumed, models built, and answers reached, between the discrete and continuous world. For example, initial workshops and lectures sessions were dedicated to exploring questions such as the ones concerning the relationship between relative rate of growth and the derivative (Q_{1vs2}) or whether similar conclusions could be drawn from the discrete and continuous Malthusian and logistic models ($Q_{1vs2.2}$). Figure 4 illustrated this parallel structure and highlights the new questions that arose from contrasting answers obtained in both discrete and continuous domains.

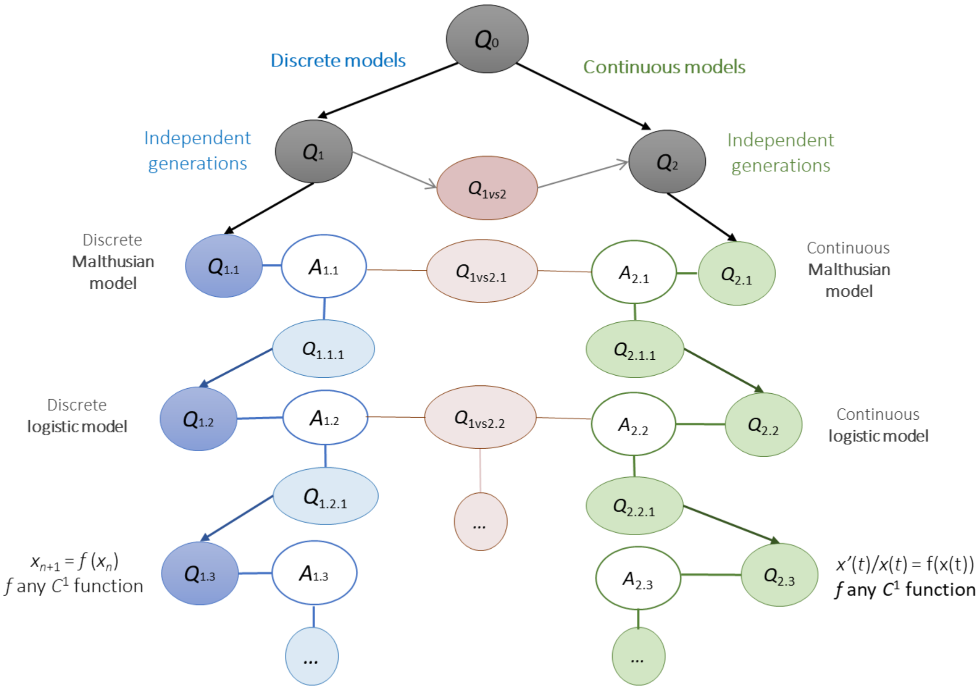


Figure 4 – QA map of the 1st and 2nd branches of the SRP with independent generation

Q_2: If we consider time as a continuous magnitude, what assumptions about the rates of growth can we formulate? What mathematical models would appear?

Q_{1vs2}: What is the relationship between the relative rate of growth and the derivative?

$Q_{2.1}$: If $r(t) = \frac{p'(t)}{p(t)} = r$ is constant, how will the network users evolve over time?
$A_{2.1}$: Construction of the Malthusian continuous model

$Q_{1vs2.1}$: Do we obtain the same conclusions from the discrete and the continuous Malthusian model? Does the constant coefficient r have the same meaning and effect on the population evolution?

$Q_{2.2}$: If $r(t) = \frac{p\prime(t)}{p(t)}$ decrease linearly, how does the population evolve over time?

$A_{2.1}$: Construction of the logistic continuous model

$Q_{1vs2.2}$: Do we obtain the same conclusions from the discrete and the continuous logistic model? Do the coefficients (K and α) have the same meaning and effect?

5.4. The 'migration' of the SRP about population dynamics in another university setting

Since the academic year 2005/06, the research group started implementing SRPs with first-year university students of business and administration degree (4-year programme) in IQS-School of Management of Universitat Ramon Llull in Barcelona (Spain). A 'mathematical modelling workshop' was introduced in the general organisation of the mathematical course. It consisted in 90-minute weekly sessions covering one-third of classroom time for students, and more than half of their personal work outside of the classroom. The instructor of the course was also responsible of the workshop sessions. These ran in parallel to the three-hour weekly lecture sessions, which included problem-solving activities. In the general organisation of a workshop, students worked in teams of 3 or 4 members, under the supervision of the instructor responsible for the course and, if possible, of a researcher who acted as an observer. The workshop focused on a generating question Q, which was broken into three subquestion. Each one of these subquestions was address in one of the three terms of the course.

The *a priori* analysis of the SRP about population dynamics was adapted to this new university setting. The first significant change for the SRP2 was about the generating question Q_0, which now centered on the evolution of users of a social network. This network, named *LunaticWorld*, was supposed to open in 2004 with 18 users, with quick growth (from 18 users in 2004 to 3143, in 2009). Analogous to the SRP1 about population dynamics, Q_0 focused on what assumption and what models could help to predict the size evolution of the users, what predictions one could make and how to test them. The first kind of models that emerged used to correspond to the 1st branch of the previously described SRP (Q_1) with models based on recurrent sequences of order 1: $x_{t+1} = f(x_t)$. It was followed, during the 2nd term, by a new development of the SRP that focused on the use of elementary functions as models to fit data and discuss criteria for its fitting (Serrano et al., 2013). The SRP2 did not include the branch associated to differential equations and systems of differential equations (as in the case of the SRP1). This omission was partly due to its absence from the official course syllabus and the lack of students' proposals in this direction. What was implemented in the 3rd term was the branch corresponding to the recurrent sequences of order $d > 1$, when x_t depends on the $d > 1$ past generation $x_{t-1}, x_{t-2}, \ldots, x_{t-d}$, which can be expressed as vector recurrent sequences with $X_{t+1} = f(X_t)$ with $X_0 = (x_0, x_1; \ldots, x_{d-1})$, which it could be alternatively described as a matrix model $X_{t+1} = M.X_t$. Figure 5 illustrates the situation as it was presented to students, wherein the social network's user group was categorized into three groups: *Basic*, *Medium* and *Premium.*

MATHEMATICAL MODELLING WORKSHOP – 3rd term

The social network *Lunatic World* has recently introduced important changes. From now on, all users will be distributed in three different groups: ***Basic***, ***Medium*** and ***Premium***. After one month, a user can remain in the same group, be promoted to another group or leave the network. Moreover, the only type of users that are allowed to invite other users to join the following month are 'Medium' and 'Premium' groups.

CASE 1 [One of the 16 cases that were distributed to the different teams]:

- *15% of 'Basic' users remains as 'Basic' one month later and 75% of 'Basic' changes to 'Medium';*
- *25% of 'Medium' changes to 'Premium' the following month and 50% remains as 'Medium'. Each 'Medium' user invites on average 4 new users that enter as 'Basic' users the following month;*
- *90% of 'Premium' remains in the same group. Each 'Premium' user invites on average 3 new 'Basic' users.*

Figure 5 – Introductory worksheet to the 3rd branch of the SRP

Some examples of the kinds of questions that were addressed in this 3rd branch are the following, which can extend the initially presented QA map (see Figure 6).

$Q_{3.1}$: How can we describe the evolution of the distribution of users in groups under the new conditions of Lunatic World network? How to use models based on recurrent equations to address this change in the hypothesis?

$Q_{3.2}$: Is it possible to transform the recurrent sequences of order $d > 1$ into a matrix model? What characteristics does this matrix $\{L\}$ have? How to iterate $X(n+1) = L.X(n)$ to forecast the future distribution of the users after some periods? [...]

$Q_{3.3}$: What are the main properties of L^n? What can we say about $\lim_{n\to\infty}\{L^n\}$?

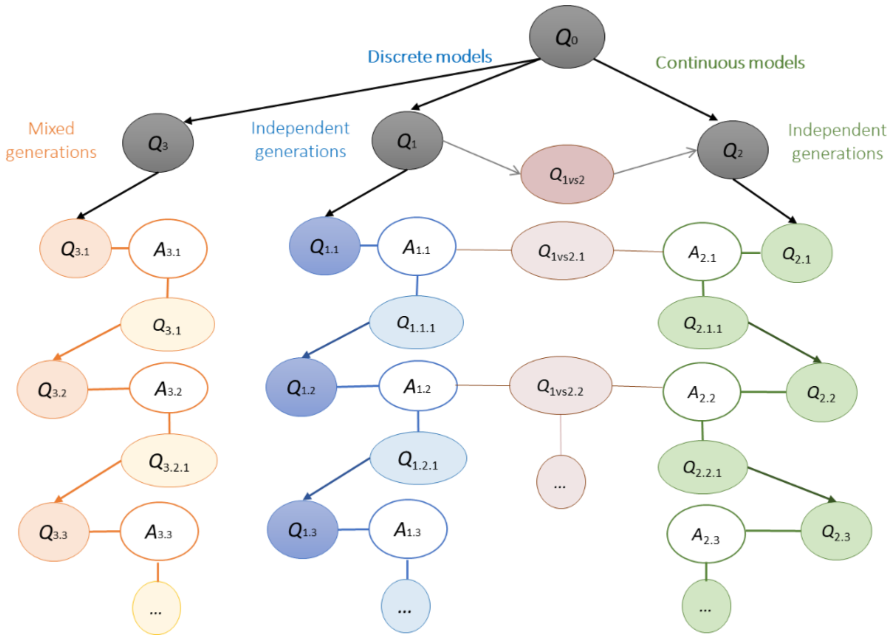


Figure 6 – QA map about mixed-generations models about the social network users

The third case refers to the SRP3, which focuses on *comparing forecasts against reality in the case of Facebook users' evolution* (more details in Barquero et al., 2018). On this occasion, the conditions for the implementation needed adaptation, as we did not have the entire course at our disposal. Instead, SRP3 was scheduled during the transition from the first to the second term within the mathematics course, targeting first-year students of the Business Administration degree and the Marketing and Digital Communities degree at the Tecnocampus-UPF University.

Once again, SRP3 was implemented in the 'modelling workshop' created for this implementation. Offered as an optional activity outside the regular course schedule, the participation in the workshop contributed with an extra point to the final grade of the subject. The workshop consisted of seven sessions lasting 1.5 hours each, with students expected to dedicate additional time outside the classroom to complete the workshop tasks.

The initial situation begins with the presentation of some selected news regarding a research project conducted by Princeton University, which anticipated that Facebook would lose 80% of its users before 2017. The generating question Q_0 was about: *Can these forecasts be true? How can we model real data about the evolution of Facebook users and forecast the short- and long-term evolution of the social network? How can we validate Princeton's conclusions?* Barquero et al. (2018) explains the design and implementation of SRP3, which encompasses three interconnected phases linked to Q_0.

The first phase centres on the exploration of data pertaining to Facebook users, while the second phase delved into identifying mathematical models, primarily based on elementary functions, that could provide a robust fit for observed Facebook users' data. The third phase focused on how to determine the most optimal and reliable model for fitting the data and employing it to provide short- and long-term forecasts of the evolution of Facebook users. Although the kind of models emerged in SRP3 primarily involved elementary functions and their derivatives, the students often used linear regression models and polynomial interpolation models. Given the accessibility of tools like Excel or GeoGebra, as some of the main media (see more details in Barquero et al., 2018), the students could explore various regression and interpolation models, either automatically provided by these programs or manually generated. Figure 7 shows this potential new branch into the previously presented QA map (in Figures 4 and 6) incorporating the regression and interpolation models.

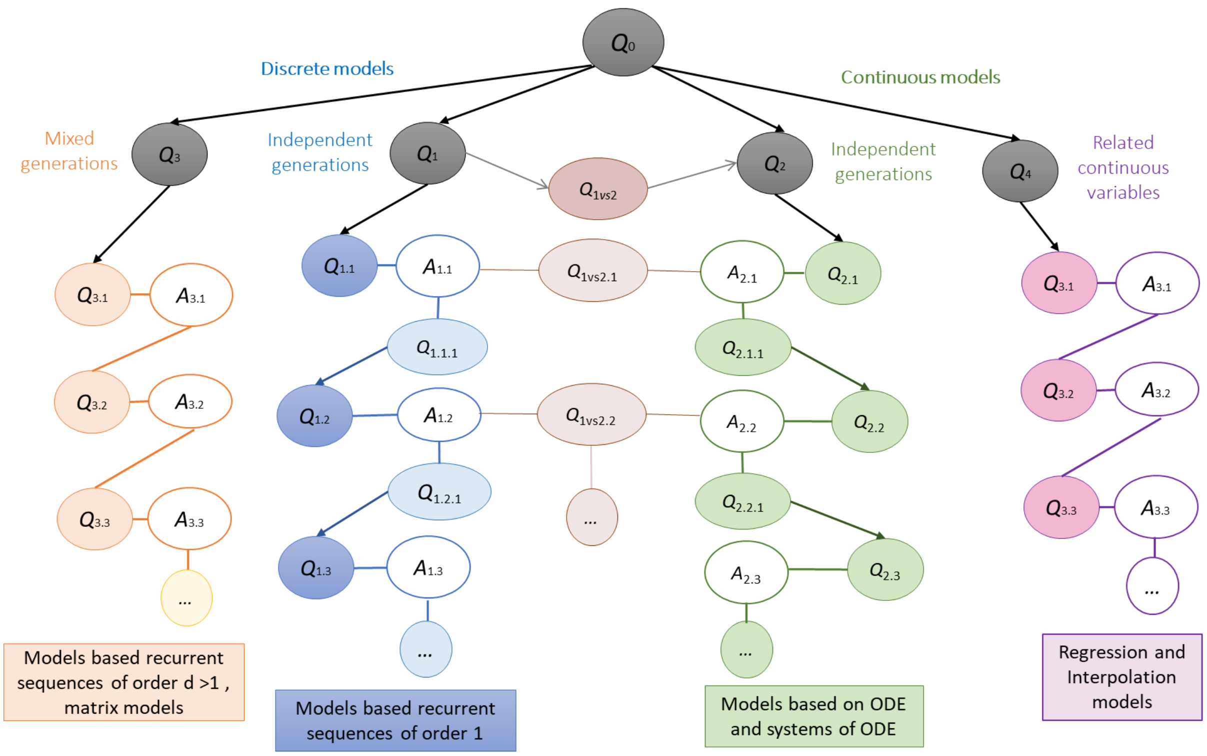


Figure 7 – QA map with models based on regression and interpolation

6. Results concerning the mechanisms fostering the dialectics of questions-answers

Through the successive implementation of the SRP, we have examined the sequence of questions arising from the generating question Q_0 about population dynamics (and its variation by considering the generating question in the context of social networks evolution). The modelling work led the students and the teachers to engage with most of the contents of the mathematics course, plus some additions (recurrent sequences, speed of convergence, graphical simulation techniques, relation between the rates of growth and derivatives, transition matrices, etc.). However, within the modelling workshop, these contents appeared in a different structure from the traditional organisation proposed by the syllabus. Instead of the traditional logic of mathematical concepts, the workshop was more guided by the 'logic of the questions to be addressed' and the 'types of models to build' that progressively appeared. After analysing these implementations, various strategies can be highlighted facilitating the dialectics of questions-answers.

6.1. Devices helping to institutionalise the dialectics of questions-answers

The institutionalization of both, the questions students dealt with and the answers they reached, was taken as a central task since the beginning of the SRP. Different teaching devices were created to facilitate this task. The first one was the *report of the week*, which served as a useful tool for students to explicitly write and formalise the questions they dealt with, the answers obtained, and new questions to follow. After each workshop session, the working teams had to write, deliver and (on some occasions) defend their team report. These reports had a fixed structure, provided by the instructor, in terms of (1) Questions addressed; (2) Answers reached in terms of the modelling process followed (hypothesis assumed, models built, answers obtained by simulation of models, techniques of validation, etc.); (3) New questions emerged from your work; and (4) Resources used.

At the beginning of the SRPs, the students had many difficulties in writing and describing their activity in these terms. But, thanks to the workshop sessions guided by the instructor and having long courses to set up changes in the didactic contract, these tasks gradually became easier. Moreover, in most of the workshop sessions, there was one team who acted as the *secretary of the week*, being the responsible of explaining their team report, based on which its content and structure were discussed. More concretely, in the first sessions of SRP1, students showed many difficulties in writing the report, often focusing only on explaining the models used and the result obtained from simulating these models. The main task for the instructor was to pose new questions facilitating the comparison of the different proposals, such as $Q_{0.1}$ concerning the hypotheses about rates of growth of the population or about the habitat capacity justifying the models, or $Q_{0.2}$ about possible relationships between the models, etc. Additionally, the secretary of the week was also responsible for presenting their advances and the rest of the groups could participate in this presentation by comparing and extending the progresses made. All this work was summarized, session after session, in the 'workshop logbook', which was shared by all the participants.

In SRP2 and SRP3, the teaching strategy shifted to incorporate two types of workshop sessions every week: teamwork sessions and working teams' presentations. During the formers, teams worked collaboratively to progress on the SRP and prepared a partial report with their advances. Then, selected teams orally defended in the teams' presentation sessions, facilitating the interaction between the teams. During the presentation sessions, the *secretary of the week* was in charge of preparing a report summarizing the key points of discussion and proposing new questions for consideration in the following sessions.

6.2. New terminology to talk about mathematical modelling

Since the first workshop session in all SRPs, an important constraint appeared soon regarding the necessity of an *ad-hoc* mathematical discourse to talk about the mathematical modelling activity undertaken and the resulting outcomes. It became evident that there was a necessity to construct a 'new' discourse around modelling, previously absent for the students. Hence, from the initial sessions onwards, it was crucial for the instructor to institutionalize specific terms to denote various aspects related to mathematical modelling, including terms such as system, variables, hypotheses, model construction, simulation, validation of the model, and other related terms.

For example, during the early stages of the workshop sessions, while the students were able to present different models, none could precisely define the variables under consideration, the hypotheses guiding the model construction, methods for model validation, or the scope of validity of the models. Consequently, the instructor began introducing new words around modelling, thereby enabling the students to develop a *didactic technology* specific to modelling. As previously mentioned, an important didactic strategy used by the instructor consisted of selecting different groups and asking them to explain and compare their modelling progress. This discussion was usually initiated by the 'secretary of the week' and followed by the rest of the working groups. As, in most of the occasions, the working groups came with different modelling 'paths', numerous questions arose regarding how to compare their work. This discussion led to questions about the modelling process itself, and students had to open and understand the different proposals according to the modelling process followed: hypothesis formulated, the interpretation of the model's coefficients, contrasting models' simulations with data, among other considerations.

7. Results concerning the mechanisms promoting the dialectics of *media-milieus*

In the preceding sections, we have introduced various elements that gradually became part of the students' *milieu*: the generating question Q_0, the derived questions Q_i, temporary answers A_i, previous students' knowledge, etc. Additionally, we have discussed different *media* through which these elements were progressively integrated into students' *milieu*. However, these were not the only elements, mainly because the workshop did not operate in isolation from the rest of university teaching devices (such as lectures, and problem sessions), and because students were encouraged to work autonomously outside the workshop. Indeed, as the students and teachers engaged in the SRP, there were different moments when the existent *milieu* was insufficient and there appeared new necessities that required stopping and extending students' *milieu* with new elements. As noted by Kidron et al. (2014, p. 158), "the media-milieus dialectic appears when considering the different kinds of general didactic gestures performed by students and teachers in the interaction with M (the *milieu*) to produce $A^{\heartsuit}$ (the final answer)". In the following discussion, we focus on underlying some basic didactic gestures that were developed during the implementation of the SRP, contributing to the progressive extension of the *milieu*.

7.1. Integration of the SRP with other devices at university level

We observed a rich interaction between the running SRP and the activities that were organised in response to the emerging needs. Indeed, in the implementations of the different SRPs, the workshop was year after year more and better integrated with the rest of the university teaching devices. Lectures and exercise-problems sessions were often used to provide students with some of the necessary tools to follow the work developed in the modelling workshop.

For instance, in the first branch of SRP1 about discrete models for populations with independent generations, the regular course typically started with a predefined syllabus covering elementary functions (linear, quadratic, exponential, etc.), as well as techniques for solving equations and inequalities through algebraic and graphical techniques. However, questions arose in the workshop that demanded the introduction of some mathematical knowledge beyond the official curricula. Consequently, the usual way to proceed was to stop and plan in the lectures and exercise-problems sessions some time to introduce these new tools (for example, when there was not a clear definition of the different rates of growth, or when it was necessary to the definition of recurrent sequences, or of specific techniques for matrix diagonalization). Then, the instructor guiding the workshop interrupted the workshop to address and work on this necessary new knowledge and tools, scheduling as many sessions as necessary in the lectures, problems, or workshop sessions.

In this sense, an important condition for mathematical modelling was to break the rigidity of the classical structure 'lectures-problem sessions-exams', which typically follows a unidirectional flow of introducing new contents and applying or exercising these contents. Instead, it was important to ensure that both lectures and problem sessions were considered during the workshop, each contributing to its development in two ways. Firstly, we could find the situation in a previously introduced answers $A_i^\lozenge$, or a mathematical work/object (W_j), which had been previously introduced to students (being part of their *milieu*), are now put into used in the modelling workshop. Secondly, as the SRP progressed, new questions Q_i or answers A_i emerged in the workshop, necessitating the introduction of additional $A_i^\lozenge$ or W_j. As a result, the lectures and problems sessions could address emerging needs opened during the development of the SRP. Thus, the ideal situation was to maintain a bidirectional relationship between all these existing didactic devices. In the various implementations of SRP1, SRP2, and SRP3, lectures and problem sessions were employed to provide students with some of the necessary tools to engage effectively in the workshop. And, vice versa, the workshop served to motivate and demonstrate the application and functionality of the course's content.

7.2. Making accessible external answers by enlarging the media

It was not only in the workshop or in the classroom university, from where new elements for the *milieu* came. Throughout the workshop sessions, students were encouraged to look for possible external answers to the questions they were addressing and (if pertinent) bring them to the workshop. Across various implementations, students looked for what existed outside, and what experts knew and said about the phenomena addressed, such as the population dynamics or the evolution of social networks (as illustrated by the SRP cases outlined in this paper). Students were asked to bring to the class some information about the phenomena we were analysing or about the models they were proposing. It was, in fact, an important part of the *weekly reports* and of the debates in the workshop. This work facilitated giving names to the models (such as the Malthusian or logistic models, in SRP1, or to the transitions matrix models, in SRP2). However, this process required meticulous study to effectively deconstruct and integrate these external insights into the dynamics of the SRP.

Another significant mechanism involved in the SRPs was the interaction with experts in other disciplines. In SRP3, as explained in Barquero et al. (2018), from the initial steps, the students were asked to look for data about Facebook users' evolution (instead of being provided by the instructors). The main *media* consulted were websites and the interaction with experts. For instance, in SRP3, a course titled 'Introduction to digital communities' organized some classroom sessions in the course to let students know what the main pages were to consult for social network data, different ways to organize data, and about pre-existing models used by experts in this field of social networks.

Furthermore, we may mention other significant mechanisms present in all the SRPs aimed at broadening the sources of information (media) accessible to students. An important step in modelling processes involves simulating models to generate new data by using models to contrast with the real ones. In this regard, simulators played a crucial role, being made available during the course for the students with specific training sessions In SRP1 and SRP2 the main technological environment was Excel. In the case of SRP3, the design of new applets that could act as media for models' simulation or model contrast or validation, was of special importance on this occasion, as researchers were collaborating with technology developers (Cinderella, Geogebra, etc.). Consequently, the lecturers suggested alternative tools, such as spreadsheets or specialized mathematics and statistics software, to facilitate students in simulating and testing their proposals.

Finally, as previously mentioned, significant emphasis was placed on the sessions dedicated to the presentations of the working teams and their weekly reports. All teams could act, as well as the instructor of the workshop, as media for the rest of the class. In this context, the designated secretary of the week undertook the difficult task of reporting on their team's progress and summarizing the advances made by the rest of the teams, ensuring that this information was readily accessible to all participants.

8. Conclusion and discussion

Over the past decades, our research group has been working on the design and analysis of several SRP at university level to promote the teaching and learning of mathematical modelling (see for an overview in Barquero et al., 2022). Thanks to this research line, we have identified many constraints that hinder their implementation, but also many desirable conditions to step-by-step progress on introducing changes in the prevalent 'paradigm of visiting works' in university mathematics education. This paper has focused on selected SRPs (implemented from 2005/06 onwards), highlighting the *mechanisms* that have been more useful and effective in fostering the integration of mathematical modelling into first-year mathematics courses. Furthermore, these experiences demonstrate the important role of the SRPs in breaking the rigidity of mathematics programmes and their teaching organization.

After outlining some of the characteristics of the SRP, and of the methodology for their design and analysis (sections 2 and 3), this paper has focused on the SRP on population dynamics to underscore the importance and utility of the a *priory* analysis of the SRP for researchers and lecturers. This analysis serves a double purpose: firstly, to explore the potential of the generating questions Q_0 and to trace the possible path to be followed in the effective implementation of an SRP. Secondly, it aims to offer researchers and educators alternative epistemological frameworks for conceptualizing mathematical activity. This *a priori* design has led us to focus on two central dialectics for the SRP: the dialectics of *questions-answers* and of *media-milieus*.

Furthermore, these initial designs have been chosen not only for their consecutive implementations within a specific university context (the one of first-year mathematics courses for natural sciences degrees), but also for their 'migration' and adaptations into other university settings, those of mathematics for business administration university degrees. Following our experience with SRP1, that have migrated from one institution to another, we have been interested in the 'ecological' invariants, that is, the conditions (also the constraints) that, independently of a change of institution, have brought light on some mechanisms that have helped that mathematical modelling take part of the regular university courses.

Therefore, not only the designs could be of interest for possible future adaptation to other university contexts, but also the results about the mechanisms that facilitate the dialectics of questions-answers and of media-milieus. We expect to open the discussion in, at least, these two main directions. On the one hand, about the transferability of the mathematical-didactic designs (as the ones synthesised in section 5). On the other hand, regarding the transferability of mechanisms aimed at enhancing the different dialectics necessary for the comprehensive development of SRPs. Both complementary aspects would contribute to the analysis of the ecology, that is, the conditions that can facilitate and the constraints that remain hindering (despite the change of institutions) the teaching and learning of mathematical modelling at university level.

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