Making UME research accessible and meaningful to the Mathematics community: A *Special Issue* commentary

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1. Introduction

This *ÉpiDEMES Special Issue* aims to offer accessible accounts of University Mathematics Education (UME) research to the Mathematics community. These accounts are intended as a mutually productive interface between the communities of mathematics and mathematics education researchers who practise the teaching of university mathematics and take an interest in questions about this teaching. The editors invited articles of two types: either articles in which researchers in mathematics education aim to disseminate results of their research to a wider audience; or, articles that report the implementation of a pedagogical and/or didactic innovation. They aspired to cover an ambitious range of themes: the secondary-tertiary transition as well as transition within university mathematics courses; teacher-researcher UME practices; the teaching of university mathematics to non-specialists; university mathematics teachers and students' practices and the links between these; pedagogical implications of the links between mathematics and its applications as well as with other disciplines in higher education; the teaching and learning of university mathematics in the digital era; and, the teaching and learning of specific mathematical topics in Calculus, Analysis, Algebra, Geometry, Discrete Mathematics, Probability and other mathematical domains.

The six papers selected for inclusion in the Special Issue address the needs of diverse student cohorts and cover many of the aforementioned issues. Isabelle Bloch and Patrick Gibel address the transition from secondary to university mathematics and describe an intervention that aims to tackle student difficulties with mathematical reasoning upon arrival at university, particularly in the mathematical domain of Calculus. Also in the area of mathematical reasoning, Zoé Mesnil and Viviane Durand-Guerrier zoom in on a topic – the notion of "implication" – that presents a challenge in the experiences of students also transitioning from secondary to university mathematics; they analyse student difficulties thereof and propose activities to help overcome these difficulties. Similarly, staying focused on a particular mathematical topic, María Trigueros and Rafael Martínez-Planell offer an overview of their own research on students' understanding of one Calculus topic, bivariate functions; through an APOS theoretical lens, they identify challenges students face when dealing with this new type of function, they stress the need for reconstruction of knowledge about one-variable functions and offer research- based suggestions to help such reconstruction. Keeping with the focus on potent pedagogical interventions, Kristina Markulin, Marianna Bosch, Ignasi Florensa and Cristina Montañola report on a Study Research Path (SRP) inquiry-based teaching design deployed in a first course in Statistics for Business Administration degree students. Zooming out to reflect on where the field is at present in relation to research about the secondary-tertiary transition, Ghislaine Gueudet and Fabrice Vandebrouck synthesise international research in this area. Finally, acknowledging that support for students arriving in universities with often very diverse backgrounds and needs has become key to their transition, Duncan Lawson, Michael Grove and Tony Croft offer a synthesis of research on mathematics support infrastructure, crucially for student cohorts whose major focus of studies is in mathematics as well as student cohorts whose focus is in other disciplines.

This brief description shows that the papers selected for this Special Issue in principle respond directly to the editors' two complementary aims: they present and synthesize research results as well as report the design and implementation of pedagogic and didactic innovations. Before commenting on each paper, in what follows (Section 2) we briefly describe the development of the field, exemplifying more particularly from developments in the two countries each one of us is based, and sketch the current international landscape of UME research. We see this as helping to situate the six papers within a wider landscape and history, and also to structure our reflection. We then reflect on the six papers of the Special Issue (Section 3) in the light of the themes highlighted in Section 2 and we conclude (Section 4) with a brief discussion of the potentialities as well as limitations – and, crucially, indications for further research – that our reading of the six papers has revealed.

2. A quick sketch of the international landscape of UME research

As outlined in the introductory paper to this *Special Issue* by Viviane Durand-Guerrier, Nicolas Grenier-Boley, Hussein Sabra, Louise Nyssen and Avenilde Romo-Vázquez, UME research now has a decades long history. A first important milestone in this history was the creation of the *Advanced Mathematical Thinking* (AMT) group at the *Psychology of Mathematics Education* (PME) Conferences in the 1980s and the resulting, highly influential volume edited by David Tall (1991) that soon became a key reference in the field. A further milestone was the 11th ICMI Study led by Derek Holton and the resulting 2001 volume (Holton et al., 2001).

In France – where one of us, Artigue, is based – research began to develop in the late seventies as attested for instance by the thesis of Aline Robert defended in 1982 (Robert, 1982). UME research has been continuously supported there by the IREM network (*Instituts de Recherche sur l'Enseignement des Mathématiques*, Institutes for Research on the Teaching of Mathematics), especially the inter-IREM University commission that fostered the collaboration between high school teachers and university teacher-researchers, between mathematics and mathematics education researchers. The Commission, for instance, supported the conceptualization and practice of "scientific debate" introduced by Marc Legrand at the IREM of Grenoble (Legrand, 1990). Research was also sustained by the creation of experimental courses, for instance the experimental Mathematics-Physics section created in 1979 at the University of Paris 7 for first-year students, on the initiative of André Revuz, then director of the IREM of Paris, and the physicist Jean Matricon. An important body of research on differentials, differential and integral processes emerged from teaching difficulties met in this section (Alibert et al., 1988). The collaboration between mathematics and mathematics education researchers within the experimental section created by Marc Rogalski at the University of Lille was also decisive for the research carried out on the teaching of differential equations in the 1980s (Artigue & Rogalski, 1990) and then in the development of research on linear algebra (Dorier, 1999). The DEMIPS[1] group recently created within the National Centre for Scientific Research should also foster this collaboration and increase the number of university teacher-researchers involved in it.

[1] https://demips.math.cnrs.fr/

From the outset, UME research in France relied on theoretical constructs developed by French mathematics education researchers, first the theory of didactic situations (Brousseau, 1997) often combined with the tool/object dialectics and the idea of games between mathematics settings (Douady, 1986), the concept of didactical transposition, and from the nineties onwards the Anthropological Theory of the Didactic (ATD) (Chevallard, 1999, 2019). This theoretical background influenced the *problématiques* and methodologies of UME research in France, as attested, for instance, by the importance given to design research through didactic engineering (Artigue et al., 2007), or the pioneering studies approaching the secondary/tertiary transition from an institutional perspective beyond the cognitive perspectives predominant at the time (Gueudet, 2008). One recent evolution is certainly the development of studies more focused on university teacher practices as shown for instance by DEMIPS.

In the UK – where one of us, Nardi, is based – also UME research began to develop in the late seventies. Important initiatives include the work at Warwick University where projects were driven by developments in psychology – for example by Richard Skemp (1976) – and by mathematics researchers-cum-mathematics education researchers – for example, David Tall (1991). The Mathematics Education Research Centre (MERC) at Warwick hosted doctoral and other studies (e.g. Alcock & Simpson's Warwick Analysis project, 2001) that focused on a systematic investigation of what the encounter with new standards of rigour entails for those entering university mathematics, especially in Calculus and Analysis. Often setting out from valuing the role of intuition that drives the work of mathematicians, the courses were distinctive for the new pedagogy they put forward: one in which small classes, collaborative learning and debate play key parts. Also exploring the new didactic contract (Brousseau, 1997) that entering university mathematics entails, was the focus of exploratory studies into university mathematics teaching and learning at Oxford University: Nardi's (1996) study of incoming mathematics undergraduates' encounter with mathematical abstraction identified the tensions experienced by students as they construct concept images in the formal context of university mathematics as well as the challenges of their enculturation into the new pedagogical context of university studies. It was followed by studies led by Barbara Jaworski (2003), conducted in collaboration with the Mathematics Institute at Oxford, of how university mathematics teachers orchestrate students' transition into university mathematics (Nardi, Jaworski & Hegedus, 2005) – and how important collaborative work is towards supporting them in doing so. Overall, the relationship between the communities of those who research/teach mathematics and mathematics education researchers has been evolving at a good – if a little leisurely – pace. Beyond those who regularly participate in mathematics education research initiatives, at least in the UK, we are seeing Mathematics Education courses being included in Mathematics undergraduate programmes and we are seeing more and more UME researchers involved in the Professional Development of those who teach mathematics. So, all in all, steps are being taken towards institutionally supported collaboration in practice, development and research. This is certainly several steps up from the beginnings of cross-community work in the 1990s when every single initiative had to start from scratch and relied heavily on personal rapport and trust with individual colleagues. Departmental (institutional) level collaborations with permanence, stability and longevity is what our aspirations should turn to. The importance of the rapprochement between the communities of mathematics and mathematics education researchers – and the mechanics of non-deficit, non-prescriptive, situation-specific, mathematically-focused collaboration through which such a rapprochement can be pedagogically productive (Nardi, 2008) – is nowhere more obvious than in the steadfast development of mathematics support infrastructure that has characterized developments in the last decade or so. Loughborough University's Mathematics Education Centre (MEC) is a flagship example of these initiatives – duly represented in this *Special Issue* by the contribution from Duncan Lawson, Michael Grove and Tony Croft.

As the two examples from our respective countries indicate, UME research has thus a rather long history. Research at university level has been the source of conceptual distinctions such as Tall and Vinner's (1981) still influential distinction between concept image[2] and concept definition or the epistemological distinction in terms of FUG[3] (Formalizing, Unifying, Generalizing) concepts introduced by Robert (1998). It has also been the source of theories based on Piagetian epistemology – such as APOS theory (1991) that Ed Dubinsky and colleagues began to develop in the eighties – and has been especially influential in the field (see the contribution to this *Special Issue* by Maria Trigueros and Rafael Martínez-Planell) or didactic practices with strong epistemological foundations such as the aforementioned tradition of scientific debate. For quite some time, the theoretical perspective of UME studies was largely developmental and dualist, with the focus on perceived differences between the intuitive and the abstract, the procedural and the conceptual, processes and objects. At an international level, and beyond the concept image/concept definition distinction and APOS theory we just mentioned, prevailing theoretical constructs have been Richard Skemp's instrumental and relational understanding[4] (1976), Eddie Gray and David Tall's procepts[5] (1994) and Anna Sfard's (1991) theory of reification and process – object duality.

The fast rising development of the UME field in the last 20 years (Winsløw, Gueudet, Hochmuth & Nardi, 2018; Durand-Guerrier, Hochmuth, Nardi & Winsløw, 2021; 9 new entries in the second version of the Encyclopedia of Mathematics Education (Lerman, 2020)) has gone hand in hand with the enlargement of questions addressed and theoretical perspectives, of methodologies and innovative practices, also connecting these with the new possibilities and challenges resulting from scientific, technological and societal evolution. As shown for instance by the *RME* special issue (Nardi, Biza, González-Martín, Gueudet & Winsløw, 2014), UME research more and more relies on institutional, socio-cultural and discursive theoretical approaches, not only on the developmental/cognitive perspectives proliferating in the AMT book for instance. This evolution influences in turn the formulation of research questions and units of analysis, moving from the understanding of students' difficulties regarding particular mathematical concepts or domains to the more global functioning of university didactic systems and institutions, and the way these condition

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[2] Defined by Tall & Vinner (1981) as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and

[3] These are central concepts in mathematics that have simultaneously the capacity to formalize, unify and generalize (FUG; Robert, 1998) properties of objects / processes. The formalization that emerges offers a simplified and compact way in which to approach mathematical properties or objects hitherto seen as disperse or distinct.

[4] Succinctly described as the difference between knowing how (instrumental) and why (relational) something works in mathematics.

[5] Procept (Gray & Tall, 1994) is an amalgam word (process and concept). Alongside Sfard's (1991) process-object duality, these two constructs highlight a duality that lies within most mathematical objects: a function *f*, for example, is both a process and an object. According to Sfard (1991), refication denotes "an instantaneous quantum leap" during which a "process solidifies into object, into a static structure" (p.20).

the learning possibilities of students, or contribute to the difficulties of the secondary/university transition. The increasing influence of the institutional and ecological perspective of the ATD, and of commognitive perspectives (Sfard, 2008) well attest this evolution.

Another important trend is the increasing attention paid to the fact that for most university students, mathematics courses are service courses. This induces specific research needs that have been marginalized for a long time (Hochmuth, 2020). Regarding domains, there is no doubt that, for decades, UME research has mainly focused on Calculus/Analysis for functions of one real variable and linear algebra, both major domains in the first university years. Here too we observe an evolution, even if still modest, with the development of research on more advanced mathematics, and also on the probability and statistics fields, so important today both scientifically and socially, and whose approach is so much transformed by technological advances. More globally, UME research should be deeply impacted by technological evolution, far beyond the attention it has paid for decades to mathematics professional software such as *Maple* or *Mathematica*. However, as demonstrated in these pandemic times, much remains to be done in this area, considering how technology impacts learning and teaching practices at large, but also the current evolution of computer sciences and the mathematical needs this evolution generates.

For a long time, UME research has focused on students. However, in the 21rst century a turn has occurred with an increasing number of studies investigating university teachers' practices, knowledge and beliefs, as had already started being the case for primary and secondary teachers more than a decade earlier. Research has investigated usual forms of university practices such as lecturing and its learning effectiveness, and also types of innovative practices, especially more interactive and inquiry-oriented practices, while highly contributing to their development. Such practices take a diversity of forms, depending on the context. For instance, in the USA, where inquiry-based learning approaches seem to be particularly developed (Laursen & Rasmussen, 2019), such approaches have developed in two ways, either driven by research, especially Realistic Mathematics Education research, or emerging from the ground itself and with little or no interaction with educational research. The two ways are substantially different. The role given to these innovative practices is also highly variable. In some cases, they permeate a whole course, but frequently they are more used as additional components. Also, many other strategies and structures have been developed to support students, as shown for instance by the support centres in the UK: all these varied developments imply that UME research has today a very rich landscape to explore and study.

All these developments have occurred alongside the development of new methodologies, moving from the predominance of student questionnaires and interviews aiming at understanding student learning processes and difficulties to a diversity of methodologies which allow researchers to take into account the institutional conditions and constraints of these processes, and to develop detailed analyses of social interactions and discourses (Artigue et al., 2007). Also, the theoretical enrichment already mentioned has led to a diversity of forms of design-based research – didactic engineering familiar to French mathematics education researchers since the eighties being only one of these and itself taking new forms under the theoretical umbrella of ATD.

Having said this, we are still at a phase where dissemination of research results to those whose practice is likely to benefit from said results is still quite limited and hardly influences university practices. Thus, the importance of interface journals such as *ÉpiDEMES* – which aspire to make research advances and promising interventions accessible to the whole community of those engaged in university mathematics education – is paramount.

In the light of these observations, the lenses through which we have looked at the papers of this *Special Issue* are: how do the papers reflect major concerns and trends of current UME research and add to current research knowledge? How do these papers make research approaches and results accessible to a wide audience, beyond the sole audience of researchers in mathematics education? How do these papers contribute to linking research and practice, through the interventions they report on or propose?

3. Reflections on the six papers of the SI

We now reflect on the *Special Issue* through the lenses highlighted – and questions listed - in Section 2. As explained in Section 1, the paper by Ghislaine Gueudet and Fabrice Vandebrouck synthesises international research about the secondary/tertiary transition – building on the much cited work by Gueudet (2008) on this topic – and offering a substantial contribution to a main focus of this special issue. Also as explained in Section 1, Duncan Lawson, Michael Grove and Tony Croft synthesise research developments concerning how universities, particularly in the UK, have responded to the need to support students' transition to university studies through reorganising institutional infrastructures to this aim. Both of these syntheses are powerful and informative accounts that target the non-specialist readership of the *Special Issue* with care and panache. The two syntheses complement each other nicely: towards the end of their synthesis, Gueudet and Vandebrouck raise the issue of mathematics support and offer examples, from the UK and elsewhere, of infrastructures put in place to address issues of transition. Embedding their examples of developments in the UK into the wider recognition – societal and economic – of what became known in the UK as "the mathematics problem", Lawson, Grove and Croft make a compelling case for what makes mathematics teaching in a support milieu distinctive and its systematic study a distinctive area of UME research.

The other four papers in the *Special Issue* report specific research studies concerning the learning and teaching within particular mathematical domains and/or concerning specific pedagogical interventions.

Isabelle Bloch and Patrick Gibel's study falls under the innovative engineering devices referred to in the article by Gueudet and Vandebrouck. Their study addresses the transition from secondary to university mathematics through an intervention that aims to tackle student difficulties with mathematical reasoning upon arrival at university, particularly in the mathematical domain of Calculus. Students are invited to become involved in problem-solving while engaged with the normal procedures of their course. Divided in small groups, their collective work involves researching the problems, writing solutions, sharing these with peers and discussing these before formally agreeing, institutionalizing, the solution of the given problems. Analysis of student productions in two exploratory situations reveals reasoning difficulties as well as conceptual difficulties in Calculus. The intervention is shown as productive for student learning outcomes: the students are starting to question the meaning of the concepts they start to engage with as they embark on their university course and their work is gradually starting to fit into the logical norms of university mathematics. The design of the proposed intervention follows closely the perspective by Praslon (2000): the transition between school and university mathematics, particularly in calculus and analysis, is not a radical break but an accumulation of smaller breaks. The intervention targets the difficulties that the students experience because of the often rapid and sudden change of the didactic contract. In the proposed intervention, students are given notable responsibility whether this concerns the flexibility between points of view on functions (punctual/local/global), the management of semiotic complexity, the choice of settings according to Douady (1986) in elaborating and formalizing proofs, or the mastery of different forms of reasoning. On the part of the teachers, their responsibilities include both carrying out the *a priori* analysis necessary to ensure authentic devolution of the problem-solving situations to the students and providing appropriate, balanced guidance to the students during the research process. Notably, this balanced guidance is particularly sensitive to allowing students genuine engagement with heuristic activity while assisting them with identifying errors and overcoming difficulties. This didactic contract is set out at the beginning of the year and the first problem-solving situation is used to establish and clarify said contract. The paper in the *Special Issue* concerns two of the seven or eight situations the students were exposed to (variation of the area of a geometrical figure in function of angle or abscissa of a point; series of squares with sides in geometrical progression). Apart from highlighting student difficulties (algebraic and other) with the problems, the analysis showcases how accessible mathematical problems can provide fertile ground for students' discovering the types of reasoning and proof they will be expected to generate as they engage with mathematics at university level. In relation to our lenses in this commentary on the *Special Issue* papers, we commend this paper for the innovative device at its heart, which has clear potential to facilitate the transition from school to university mathematics and mitigate against dropout of disenfranchised students.

Also in the area of challenges faced by students as they encounter the requirements of formal mathematical reasoning upon entry in university studies, Zoé Mesnil and Viviane Durand-Guerrier zoom in on one particular topic, the notion of implication in the context of first order predicate calculus, and propose a balance between formal and less formal approaches. Their proposition stands on the shoulders of a confident, well-informed account of theoretical considerations which factor in syntax, semantics and pragmatics (Durand-Guerrier & Dawkins, 2020). We commend the way in which these theoretical considerations are put into direct pedagogical purpose: to identify and analyse students' difficulties (in one case, concerning statements containing the expression "two by two"; in another, proving the convergence of the sum of two convergent sequences). The activities proposed by the authors right after emerge naturally from their analysis of student difficulties and are underpinned by aforementioned balance between more and less formalism (perhaps more directly applicable to students in the French context, already partly familiar with some of this formalism from their high school studies). While the authors focus more on the design and principles underpinning the proposed activities, and less on their trial and evaluation, their proposition serves their stated purpose well: "to make readers aware of often unnoticed didactic phenomena, which call for epistemological vigilance (Artigue, 1991) in order to question our practices as teachers and to clarify some of the expert approaches that guide mathematicians in problem-solving situations" (Mesnil & Durand-Guerrier, this special issue, p.24). We note the authors' attention to proposing activities that fit well with existing course structures and contents and we commend their holistic take on the relevance of their initiative also outside purely mathematical courses (they mention the examples of computer science, teaching logic and teaching in multi-lingual contexts). In relation to our lenses in this commentary, we note the rarity of contributions – and the scarcity of experts – within UME in this area. The paper brings to light analyses of syntactic, semantic and pragmatic perspectives for language in the calculus of predicates, particularly essential at the university level. The paper also sits neatly alongside Durand-Guerrier & Dawkins' (2020) contribution to the *Encyclopedia of Mathematics Education*. We commend the effort put by the authors into reaching out to readers outside the community of UME researchers and for doing so in a way that does not sacrifice conceptual integrity. We also commend the effort to flesh out the proposed activities from research findings and to present the integration of these activities into regular teaching structures as reasonably easy. This is no small feat for the really challenging area of logic.

Similarly, staying focused on a particular mathematical topic, bivariate functions, María Trigueros and Rafael Martínez-Planell offer an APOS-theory inflected overview of their own research of fifteen years in this area. Starting from identifying challenges that students face when dealing with this new type of function, they stress the need for reconstruction of knowledge about one-variable functions and offer research-based suggestions to help such reconstruction. Underpinning their proposition is the claim that students cannot generalize from their knowledge about one-variable functions and that the setting of different foundations – semiotic and other – is necessary for students' introduction to bivariate functions. Their work is deeply embedded into APOS theory. As we highlight in Section 2, this is a theory that has been tightly linked to UME research all the way since the eighties. Of special relevance to the lenses through which we write this commentary, is the link of APOS studies investigating the students' conceptual development in many different mathematical domains with the didactic methodology known as the Activities - Classroom discussion - Exercises (ACE) cycle in which student group work alternates with class discussion. The *Special Issue* paper shows the connection between these two facets of APOS research – exploration of student difficulties and genetic decompositions of key mathematical topics converted into activities applicable in class. The paper also demonstrates the reflexive relationship between the two: in the light of classroom trials, genetic decompositions are also revised. Apart from the longevity of the research programme presented compactly in the paper – which notably involves several research cycles on the calculus of bivariate functions – a commendable feature of the showcased work is the relatively advanced mathematics this work deals with and its importance in numerous service courses. We alert the readers of this paper though that some familiarity with the genetic decompositions that underpin the proposed pedagogical activities is necessary and will strengthen the appreciation of the proposition put forward by the authors.

Keeping with the focus on potent pedagogical interventions, Kristina Markulin, Marianna Bosch, Ignasi Florensa and Cristina Montañola report on a Study Research Path (SRP) inquiry-based teaching design deployed in a first course in Statistics for Business Administration degree students. Following a summary of the course's coming to be and its design, Markulin and colleagues offer analysis that embraces teacher, student and researcher perspectives. Crucial to their ATD analyses is the observation that course design, implementation and evaluation cannot be disentangled from the university programme and overall institutional structures in which it is embedded. As with the Trigueros and Martínez-Planell's paper, there is evident reflexivity in the authors' account of how course components feed into the SRP and how course content evolves towards meeting students' professional needs. On the way, all parties, researchers and practitioners alike, learn about what inquiry-based pedagogy entails. As with some of the first examples of SRP initiative in the UME context – for example, based on Barquero's doctoral study on incorporating Mathematical Modelling activities (Barquero, Bosch & Gascón, 2013) – there is value and significance in SRP efforts to address student needs in teacher education or other service courses. Here the focus is on Statistics, a mathematical domain largely underserved by UME research, especially in comparison to more traditional domains such as Calculus, Real Analysis or Linear Algebra. Notably, the authors engage with another area that is relatively under-researched: the evolution of digital tools – such as the R software – for the teaching of Statistics. Navigating across the substantial and dense account of ATD theoretical constructs in the paper (chronogenesis/ mesogenesis/ topogenesis structure of SRP analysis, Chevallard's paradigms of "visiting works" and "questioning the world", and how SRP supports the shift from the former to the latter thus overcoming the obstacle of applicationism), before formally describing the connection of SRP to didactic engineering, may pose a challenge to non-specialist readers. Having said that, rewards await in the detailed, ecology-rich description of the course that follows. And, once again, we have a remarkable example of productive collaboration between those who do and those who do not do research in UME. The feeling that developing and implementing such a course is certainly valuable and rewarding for those involved is palpable. What is also evident is that this kind of work requires strong investment and time, making the ecology of such a course fragile. As the authors note, this is typical of other SRPs too.

4. In conclusion: pertinent foci in current and future UME research

With the riches of the six papers now showcased, we now conclude with observations on issues that we see as pertinent foci in current and future UME research. We navigate through the *Special Issue* papers with these foci in mind.

The surge of active/participatory learning approaches. As mentioned in Section 2, an important line of development in current research regards the development of active/participatory learning approaches. Much remains to be done, however, as higher education faces constraints that do not favour such evolution. Among the strategies adopted, a number now fall under the Inquiry-Based Education (IBE) banner. Two papers in the Special Issue in particular emphasise practices that place inquiry at the heart of learning: the papers by Bloch and Gibel and by Markulin, Bosch, Florensa and Montañola. They show us a phenomenon already identified in primary and secondary education, namely the variety of practices covered by this banner. It can be linked to the theoretical choices that guide the design (Artigue & Blomøj, 2013). Here the respective choices of TDS and ATD lead to different didactic constructions. But diversity is also linked to the place and status given to these alternative practices in the overall organisation of teaching. Often, for reasons of feasibility, they are set up in ancillary devices, as is the case for the first paper, or concern a component of the project type, of the main teaching, as is the case for the second one. They can be initiated at even more local levels, as shown in the European project PLATINUM[6], which brings together teams from eight universities on this theme. In order to overcome the fragmentation that may result, to facilitate the capitalisation of the results of these innovations and research, it is important to develop tools that help to organise this diversity and to encourage collective work and exchanges between communities. This is precisely what the PLATINUM project partners are doing. Furthermore, what this set of contributions makes clear is that alternative practices, likely to inspire practices are not limited to those that come under the IBE banner. The teaching based on the ACE cycle underpinned by the APOS theory, the activities and interactions implemented in support centres in the UK are excellent examples. In fact, there exists already a large number of resources that need to be better known and exploited.

The fast rise of digital experiences in mathematics learning and teaching. In the papers selected for this *Special Issue*, digital technologies and experiences are not a focus point. High school technology such as GeoGebra is mentioned in the papers by Bloch and Gibel, Trigueros and Martínez-Planell. These authors show how GeoGebra supports visualization and the students' heuristic activity. However, in their synthesis, Gueudet and Vandebrouck rightly point out that university teachers tend more to banish the high school digital tools familiar to students than to integrate them into their practice. The paper by Markulin, Bosch, Florensa and Montañola provides a different vision more in line with what can be expected from university practices. Not only the software R used by students bridges with the practices of university mathematics researchers, but also in the SRP, digital resources are used for accessing information and resources, collecting data, communicating. In their review, Gueudet and Vandebrouck acknowledge the diversity of tools available both to support students' mathematical activity and to encourage their collaborative work, then focus on two modalities whose interest has been well documented by research: the online exercise base WIMS and an online bridging course mobilising various technological resources for diagnosis, visualisation and the interactive resolution of exercises. Furthermore, as Gueudet and Vandebrouck stress, technological resources are not self-sufficient: the effectiveness of their use depends a lot on the quality of the didactic scenarios in which they are used. For sure, UME research needs to focus more systematically on the potentialities of digital technologies, and on how students and teachers combine a diversity of professional and social digital tools with a diversity of roles in their practices. The current pandemic situation has made this need even more evident and even more urgent.

The much needed pedagogical support for and preparation of university mathematics teachers. For a long time, this preparation was almost non-existent, as the11th ICMI study lamented. The situation is changing, as Jaworski (2020) reported in the corresponding article in the *Encyclopedia of Mathematics Education*. Initiatives indeed are starting to emerge in various countries even if they remain fragmented and sporadic, in spite how desirely needed they are. Needs are not limited to the preparation of new university teachers who have to deal with increasingly large and heterogeneous learners, but must also be thought of in terms of longer-term professional development and career progression. Work is starting to emerge that shows the growing interest for

[6] https://platinum.uia.no/uia/

such professional development, especially through involving university teachers in innovation and/or research projects where mathematics and mathematics education researchers collaborate, where communities of inquiry are formed. In this area too, it is important to share progress.

The broadening scope of UME research into the study of mathematics teaching and learning within the natural and social sciences. Last but not least, a theme we would like to address in our concluding thoughts is that of UME research focusing way beyond the studying of teaching and learning of mathematics for mathematicians and mathematics teachers. As with the previous theme, this has long been under-represented in UME research, although it concerns a large number of students and teachers (Artigue, Batanero & Kent, 2007; Hochmuth, 2020). Often, institutional structures - in France, for example, the separation between public universities and the elite of “Grandes écoles” – do not favour the development of didactic research in this area. However, here too the needs for development and research are very important. They are not limited to the training of future scientists and engineers, at a time when increasingly easy access to large amounts of data, artificial intelligence and technological advances more generally are substantially modifying professional practices and mathematical needs. As for the evolution of teaching practices, it is undoubtedly important not to neglect the innovative practices that have developed or are developing in the field, often independently of our research, and to associate the creators of these innovations with our reflections and work. The collaborations to be developed here are not limited to mathematics and mathematics education researchers. This was clearly demonstrated by the 20th ICMI Study conducted jointly by the International Council for Industrial and Applied Mathematics (ICIAM) and ICMI (Damlamian, Rodrigues & Sträβer, 2013).

As these reflections show, the didactic research needs raised by UME are important and diverse, concerning students, teachers, and the university organizations and institutions that condition teaching and learning alike. These needs are constantly evolving as a result of scientific and technological advances, institutional and social developments, and the crises we are experiencing, renewing both the questions and the means of addressing these questions. UME as a field of didactic research is very dynamic today. It relies on patiently elaborated achievements but also constantly develops conceptualizations and strategies, tests its achievements in action and, in return, is nourished by these and other innovative practices. It develops national and international collaborations. The articles selected for this *Special Issue* reflect this well, and give a small yet noteworthy sampler of this vitality.

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