The need for reconstruction: students' learning of the calculus of bivariate functions

María Trigueros & Rafael Martínez-Planell

Abstract. This article presents a review of our research on students' understanding of the calculus of bivariate functions. It summarizes findings from studies conducted during 15 years of research on the topic with the aim of disseminating our overall results in an accessible format. The results discussed underscore the new challenges that students face when dealing with this new type of function and suggest that the belief that students can easily generalize from their knowledge of one-variable functions is not sustained by research, so the different foundational notions necessary for the context of bivariate functions need to be considered explicitly during instruction. We include research-based suggestions that have practical value for teaching these functions, and update the state of research in this important area of the didactics of mathematics, which makes the need for further research apparent.

Keywords. APOS, calculus, multivariable calculus, two-variable functions, bivariate functions

Résumé. Une revue de nos recherches sur la compréhension par les étudiants du calcul des fonctions bivariées est présentée. Il résume les résultats des études menées tout au long de 15 ans d'activités de recherche sur le sujet. Son objectif est de communiquer nos résultats de recherche dans leur ensemble de manière accessible. Les résultats dont nous discutons soulignent les nouveaux défis que les étudiants rencontrent lorsqu'ils traitent ce nouveau type de fonction, et suggèrent que la croyance selon laquelle les étudiants peuvent facilement généraliser à partir de leur connaissance des fonctions à une variable, n'est pas soutenue par la recherche, de sorte que différentes notions fondamentales qui sont nécessaires dans le contexte des fonctions bivariées, doivent donc être explicitement prises en compte lors de l'enseignement. Nous incluons la présentation de suggestions, fondées sur la recherche, qui ont une valeur pratique pour l'enseignement des fonctions bivariées et peuvent éclairer l'état de la recherche dans ce domaine important de la didactique des mathématiques, ce qui rend évident la nécessité de poursuivre des recherches.

Mots-clés. APOS, Calcul, Calcul multivariable, Fonctions de deux variables, Fonctions bivariées.

Contents

1. Introduction

Multivariable calculus is an important tool for modeling a wide variety of phenomena that appear in science, mathematics, engineering, economics and other disciplines. Although one-variable functions and analysis have received much attention from researchers in mathematics education, the same is not true in the case of multivariable functions. The earliest research we found on this topic was published in the 1990s by two researchers. Tall (1992) discussed the particularities of the derivatives of two-variable functions, while Yerushalmy (1997) described results obtained while working on two-variable functions with secondary school children. For many years, these papers and the topics they addressed received little attention. However, in the last fifteen years greater interest has arisen in the calculus of multivariable functions, especially two-variable functions (for a survey, see Martínez-Planell & Trigueros, 2021).

We have been working on students' understanding of two-variable functions continuously for the last fifteen years. In this paper we reflect on what we have learned and what needs to be done. We consider this reflection important to promote more studies from different points of view and to provide teachers who give courses on multivariable calculus valuable information on research results concerning students' learning in this field of mathematics We begin by briefly discussing the theoretical framework that has guided our research. After that, we examine what we know today about students' understanding of geometrical aspects and other basic properties linked to two-variable functions, including their definition. We then consider students' learning of differential calculus before finishing with a brief introduction to what we have studied to date about the integral calculus of these functions. We focus on questions such as, how do students understand two-variable functions?; are the notions about real-valued functions directly generalized into the context of multivariable functions?; and how do students use what they have learned earlier? We also identify important challenges that students face when encountering these new functions, such as the expansion of the range of possible contexts where algebraic expressions are interpreted, visualization of 3D mathematical objects, the relation between students' 3D mathematical schemas and their intuition about the world we live in, and the reinterpretation of some basic notions related to one-variable functions in the new realm of two-variable or multivariable functions. Throughout the article, the reader will notice that there are still many open questions that researchers need to address and where the aid of experienced teachers can contribute to deepening the knowledge we have today.

How do we learn mathematics? Learning is a complex phenomenon. Multiple important factors influence what we learn and how we do it and mathematical learning is no exception. Several theoretical frameworks have been proposed to understand the diverse facets of learning.

Our research has approached mathematics learning through a cognitive theory based on Piaget's epistemology (Piaget, 1970) adapted by Ed Dubinsky and his collaborators in the Research in Undergraduate Mathematics Education Community (RUMEC). This theory, called APOS, proposes constructs that make it possible to describe the learning of advanced mathematics (Arnon et al., 2014; Dubinsky & McDonald, 2001). The main theoretical structures involved are Action, Process, Object and Schema, hence the acronym APOS.

2. A brief introduction to APOS theory

APOS theory holds that the construction of mathematical knowledge for any concept or topic begins with a generic student performing Actions on a mathematical Object that she or he has already constructed. Examples of Actions include comparing, calculating, following the steps of a given algorithm, or representing, among others; for example, calculating the slope of a line using the formula $\frac{y_2-y_1}{x_2-x_1}$, using formulas to calculate partial derivatives, or comparing similarities and differences when graphing two-variable functions and one-variable functions. As students reflect on their Actions, they may be interiorized, which means that they develop the ability to carry out Actions without external help, and so may skip, justify, or generalize them. Under these circumstances it is considered that the student shows the construction of a Process. For example, a student who can flexibly think of slope as independent of representation, including as a rate of change, shows the construction of slope as a Process; a student who can generate the dynamic imagery necessary to think of a partial derivative at a point as the slope of a tangent line obtained as a limit of slopes of secant lines, and equivalently, a limit of difference quotients, shows behavior consistent with the construction of a partial derivative at a point as a Process.

The construction of Processes is also demonstrated when the student can reverse the Process. Different Processes can be coordinated into a new Process. When the student needs to perform a new Action on a Process, and is able to do so, he or she shows that the Process has been encapsulated into an Object, demonstrated by her/his possibility to perform Actions on it. For example, a student who can apply Actions on the notion of slope to construct a Process of a partial derivative, or a Process of the directional derivative –as suggested in Figure 5– shows evidence of having encapsulated slope as an Object. A student who demonstrates the ability to compare partial derivatives at a given point in terms of their geometric and algebraic representations and to relate them to the total differential at that point –as suggested in Figure 6– gives evidence of the construction of a partial derivative at that point as an Object. Objects can be de-encapsulated into a Process when needed, and new Actions can be performed on the Objects constructed.

A Schema, finally, is an encompassing structure made up of Actions, Processes, Objects, and other previously-constructed Schema, as well as the relations among them (Figure 1). When a student faces a mathematical problem, she or he uses a Schema to address it. As the student constructs new knowledge, the Schema develops through the assimilation or accommodation of new structures and/or through the construction of different types of relations among the components of the Schema.

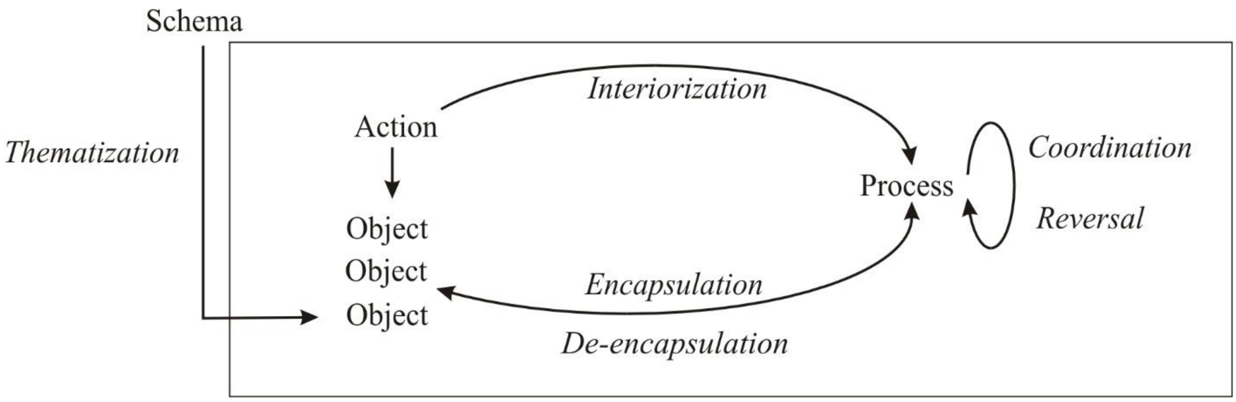

Figure 1 – APOS structures and mechanisms.

Another important idea in APOS theory is genetic decomposition (GD). A GD is a model, expressed in the terms of the structures and mechanisms of APOS, of how a student may construct a specific mathematical notion. A GD is not meant to be unique. Different researchers, or even the same one, may propose different GDs. It is important that a GD be tested with student data. To test a GD, one may start by researching what students who have studied a topic have learned and then comparing the constructions they show to those proposed by the model, or by designing activities to help students carry out the constructions proposed in the GD. After implementing the activities, experimental data is obtained from the students. The analysis of that data is expected to result in a revision of the GD that takes into account unexpected constructions shown by the students that are not included in the model and the deletion of proposed constructions that students do not seem to need. At that point, activities are designed or revised to reflect the refined GD. This opens the door to further research cycles to continue testing the revised GD. Cycles may continue until it is judged that a certain GD does not merit further revision. At that moment, the GD can be utilized to describe how most students may construct the mathematical notion, or notions, of interest, so it will also serve as a guide for instruction. In what follows, we discuss what we have learned about the learning and teaching of bivariate functions from our research using APOS theory.

APOS theory includes a didactic methodology known as the ACE Cycle (Arnon et al., 2014) that is designed to foster students' reflections and their construction of knowledge.

2.A. How can we generalize what we know about one-variable functions to bivariate or multivariate functions?

Our research has shown that it is not easy for students to directly generalize what they know about one-variable functions and then apply it to bivariate functions. Teachers must work with students to develop the tools that we have found necessary for them to deeply understand these functions.

One important and unexpected finding of our research is that constructing a coherent 3D Cartesian space Schema requires explicit attention during instruction. Results from later studies clearly showed that the time devoted to helping students understand the description of points, curves, and surfaces in 3D space contributes greatly to their understanding of bivariate functions, their graphs, and their properties. One particularly important concept that students who are trying to understand 3D space need to construct is a fundamental plane. We consider that what we and others call a "fundamental plane" is a plane in the form $x = c$, $y = c$, or $z = c$, where c is a constant in three-dimensional space. Students must be able to interpret equations in 3D space, and fundamental planes play a key role in this process. Through cycles of research that we will describe later in detail, we have found that an effective method for doing this when given a bivariate function is to fix one of the variables so that the problem is reduced to one in two dimensions where students can apply what they have learned previously. At that point, they can allow the fixed variable to change. Setting a variable constant in a given analytical expression for a surface is equivalent, from the geometrical perspective, to intersecting a fundamental plane with the graph of the surface. The plane can then be moved along an axis by changing the value of the constant, c. Once students have performed and reflected on these Actions using diverse fundamental planes and surfaces, they can construct a Process that allows them to imagine graphs of two-variable functions, cylinders, and other surfaces. These actions are essential. Fundamental planes play an important role throughout courses on the analysis of bivariate and multivariate functions. In terms of APOS theory, this is reflected in the role they play in the genetic decomposition model for bivariate functions.

Our studies of bivariate functions have advanced through several research cycles conducted in terms of APOS theory, where we have refined the GD model and designed and redesigned activities to teach the topics usually covered in multivariate analysis. We now describe these research cycles with our findings and the need that arose to reconsider some of our initial hypotheses until we could observe in the data obtained that most students showed evidence of having learned the topic of interest.

2.B. First research cycle on the geometric aspects and definition of two-variable functions

We began our research by analyzing whether students who had taken a lecture-based course on multivariable calculus had constructed the structures predicted by the GD, and if they did not show some of those constructions, asking ourselves if they were necessary for learning or if other constructions that we had not considered earlier needed to be included. Our findings evidence that in order for these students to succeed in explaining what a bivariate function was, in describing lines and planes in Cartesian 3D space, in recognizing the difference between transversal sections and projections, and in recognizing equations for certain geometrically-represented surfaces, or *vice versa*, students required a solid construction of a 3D space Schema (Trigueros & Martínez-Planell, 2010). In addition, students who recognized fundamental planes and were more confident in their understanding of space were able to conduct Actions on fundamental planes to solve problems related to two-variable functions. We concluded that constructing a fundamental plane as an Object was important.

Another interesting finding can be illustrated by the following example. When asking students to graph curves as the set of points that satisfy two equations at the same time –for example, $z = x^2 + y^2$ and $y = 2$ – we found that students gave different meanings to the resulting equation depending on the geometric context they assumed for it (Figure 2). After substituting 2 for y in the equation, $z = x^2 + y^2$, some students obtained $z = x^2 + 4$, and observing only the variables x and z assumed that the graph was a parabola on the xz plane. Other students, recognized that they must have $y = 2$, so they interpreted the context as restricted to the given fundamental plane, which led them to obtain a parabola on the plane $y = 2$. In situations where there was no restriction on the value of y, students would need to recognize that in 3D the equation $z = x^2 + 4$ refers to a cylinder. The context of an algebraic equation depends on the given problem situation and must be assumed by the student. The tool that the student possesses to determine the appropriateness of an assumed context is geometric visualization. Hence, in our studies, we learned that visualization plays a very important role in allowing students to decide on the appropriate context for an algebraic equation and, therefore, on the possibility of graphing and interpreting two-variable functions and other surfaces.

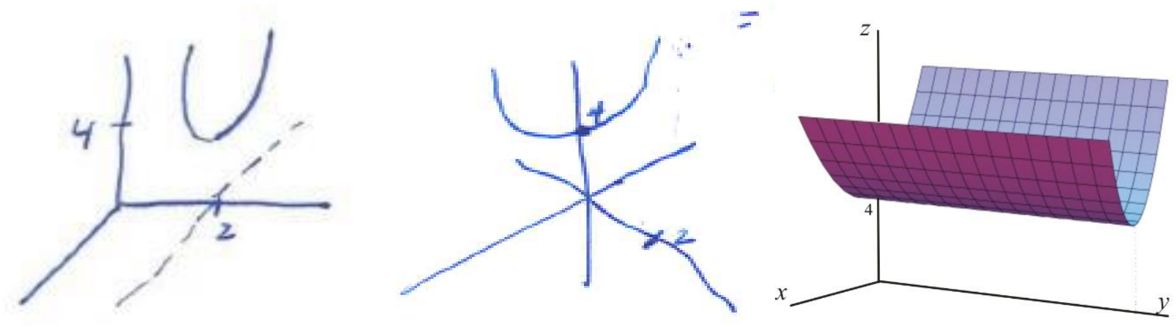

Figure 2 – Possible interpretations of $z = x^2 + 4$.

We further found that the language and gestures used to describe mathematical situations can cause conflicts with everyday language use. For example, some students interpret that the phrase "lift a curve" results in a surface, or that the instruction "cut this surface with this plane" produces two pieces of surface. Unless they are discussed explicitly with students, certain phrases used with the intention of generating a specific mathematical interpretation can conflict with students' interpretations. This can be explained as a conflict between mathematical Schemes and our intuitions of space. For example, when the researcher asked some students to "lift this circle up to $z = 3$", referring to the circle on the *xy* plane in Figure 3, s/he expected them to produce only the circle at the height $z = 3$, but the students explained that they would obtain the cylinder surface in the figure.

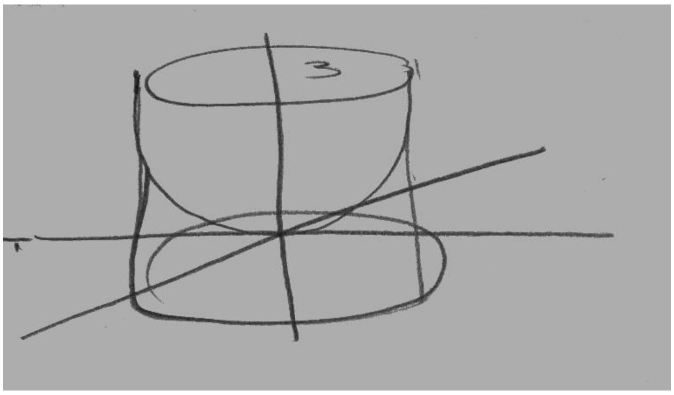

Figure 3 – A student's view of points in R^2 that satisfy $x^2 + y^2 = 1$ and are assigned a height of 3.

Based on our experience, we can suggest that teaching students about objects in 3D space requires a careful use of language and that explicit attention must be given to the construction of an R^3 Schema. For example, we have successfully used the embodied language "move forward/backward" for "x increases/decreases", "move right/left" for "y increases/decreases", and "move up/down" for "z increases/decreases", consistently during a semester, while also consistently drawing the positive x axis as if it were coming out of the blackboard ("forward"), the positive y axis pointing to the right, and the positive z axis pointing up. We also realized that the use of teaching manipulatives can help greatly in this endeavor. Pedagogical strategies of this kind can help students relate their mathematical 3D Schemas to their intuition of the space they live in.

When analyzing students' descriptions of the domain and range of bivariate functions we observed that some of them considered that "x is the domain, y is the range", and that others needed to graph the function in order to find its domain. Moreover, when considering functions with restricted domains such as $f(x,y) = x^2 + y^2$ in $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, they still needed to graph the function first in order to respond, or insisted that the domain must be obtained in some way or another from the formula for the function, $f(x,y) = x^2 + y^2$, even though the domain is given explicitly as $-1 \leq x \leq 1, -1 \leq y \leq 1$. We also observed that students struggled to describe what the unicity of the z value meant, to give a correct definition of bivariate functions, and to relate a contour diagram to graphs of the function (Martínez-Planell & Trigueros, 2012).

Our results led us to conclude that bivariate functions are not a direct or expanded generalization (Harel & Tall, 1989) of one-variable functions. Rather, students need to reconstruct their knowledge about one-variable functions in order to construct the notion of bivariate functions.

A replication study based on our first research cycle was conducted independently in Turkey by different researchers (Şefik & Dost, 2020) using essentially the same interview instruments as we had. Their results were similar and the authors basically coincided with our conclusions. This increases our confidence that the results are not context-dependent and can be generalized to other educational institutions, thus giving additional assurance of the implied generality (Aguilar, 2020) of the results that we have described thus far.

2.C. Second research cycle

The initial analysis of students' responses revealed the need to refine our proposed GD. We did so by considering what we learned in the first experience, then designed activities for students to work in teams with the goal of helping them reflect on their Actions so they would be able to perform the constructions we considered necessary to understand two-variable functions.

APOS theory, as explained above, includes a theoretical framework, a research methodology, and a teaching methodology. The latter one, called the ACE cycle (Arnon et al., 2014), involves a procedure to teach mathematical topics through the use of activities and collaborative work in teams, guided by teachers in the construction of the structures described in the GD. The teacher poses new questions related to those raised by the students and encourages the students to reflect on their work. After a certain time, the teacher calls the students to a whole-group discussion where teams present their work and new questions that arose while they were performing the assigned activities. This setting offers students new opportunities to reflect on what they have done. As part of the discussion, the teacher and students together may define some concepts or work on proofs of problems according to the issues that have been discussed. These two activities –student group work and class discussion– are repeated throughout the course and the teacher proposes some new activities or problems from texts or notes for students to work on as homework.

The second cycle (Martínez-Planell & Trigueros, 2013; Trigueros & Martínez-Planell, 2015) included research in two course sections on multivariable calculus. In one, the teacher used APOS-based activities and the ACE cycle to work with students (APOS-section), while the other adopted a lecture-based approach (lecture section). Research was conducted with both groups. Students were selected by applying the same criteria in both groups. Interviews were recorded and the researchers analyzed and discussed the data, and negotiated differences in their analyses until they were able to reach an agreement.

Results showed that most students in the APOS section manifested a better understanding of fundamental planes and their use in graphing or interpreting bivariate functions, while those in the lecture section showed similar constructions to those found in the first research cycle. Moreover, new observations arose from the data. The following aspects were found to play very important roles that we had not considered in the refined genetic decomposition: the importance of students understanding the role of free variables (e.g., $f(x,y)=x^2$; $f(x,y)=x\sin y$ when $x=0$), the fact that familiar algebraic expressions ($x^2+y^2=1$, $z=x^2$) could inhibit the generalization of the properties of one-variable functions to those of two-variable functions, the importance of the students' ability to perform and recognize treatments applied to functions presented in specific representation registers (Duval, 2006) in order to better interpret and understand (e.g., some students could graph $z=x^2$ but not $f(x,y)=x^2$), the convenience of reviewing the topic of "transformations of one-variable functions" so the students could use transformations when needed, and the persistence of difficulties in the use of fundamental planes to graph cylinders. Therefore, although the students in the APOS section clearly showed a better understanding than those in the lecture section, the results indicated that the GD and the activities designed with it needed to be refined to include these recently discovered constructions. We decided to conduct a new research cycle once the required changes had been made.

2.D. Third research cycle

Two sections as described above –APOS and lecture– were used in this research cycle as well, but the data obtained were astounding as the APOS section students performed much better than those in the lecture section, who showed results quite similar to those of the previous research cycles. We concluded that it is possible to reconstruct the one-variable function notion and apply it to bivariate functions using the activities designed to successfully promote students' reflection on their Actions. Most students showed that they had constructed a Process conception of bivariate functions and some had even constructed an Object conception of these functions. A didactic approach like the one we designed can help students construct two-variable functions by reflecting on specific Actions and Processes to develop a 3D space Schema and a fundamental plane as an Object that can be used to interpret describe, and graph two-variable functions. The difficulties faced in the second cycle were overcome and the APOS section students were able to work seamlessly with curves, cylinders, and bivariate functions in general. They were also able to justify their work and showed the construction of a dynamic image of functions that they were able to use independently of the representation register used. We thus validated the GD and activities that were designed and applied (Martínez-Planell & Trigueros, 2019).

2.E. Future research on bivariate functions

Although we were satisfied with the results of the third research cycle, the APOS section in that cycle was taught by a professor who was familiar with APOS theory and the ACE teaching cycle, so we asked if it would be possible to obtain similar results in sessions conducted by other teachers in different conditions. We decided to conduct a reproducibility study (Sánchez, 2020) with an interested professor-researcher at a university in Iran. He used all the material designed for the third research cycle. The results were encouraging (Borji et al., 2022), although there are substantial differences that need to be analyzed in greater depth.

We are also in the process of integrating digital technologies, specifically GeoGebra software, to help students visualize 3D space, fundamental planes, and two-variable functions. We will adapt the activities designed so far to this software, implement them in the classroom, and conduct research to determine if they influence students' understanding and to what degree.

3. Learning bivariate functions in differential calculus

Encouraged by these results, we decided to continue our research on the calculus of two-variable functions by following a research process similar to the one described herein. Once again, we focused on the question of students' ability to directly generalize what they know from real one-variable function differential calculus and apply it to two-variable functions.

We consulted previous studies on students' learning of differential calculus of bivariate functions in which the authors used theoretical frameworks distinct from APOS Theory (Yerushalmy, 1997; Weber, 2015). Both studies focused on the generalization of the rate of change notion for these functions. The main problem involved in dealing with differential calculus of bivariate functions is that for one-variable functions there is only one rate of change at a given point, but in bivariate functions the rate of change at one point in the domain is not unique. Direction plays a key role in this case. Both of the earlier studies found that even good students who were able to construct rates of changes in the x and y directions for a bivariate function struggled to construct what they called "the" rate of change which they hoped would exist (Weber (2015) called this the "two change problem"; see also Moore-Russo, Conner & Rugg (2011)). The students in those studies showed a tendency to look for an algebraic solution to the problem but were unable to express exactly what was to be generalized.

In our studies, performed with the collaboration of Daniel McGee (2015, 2017, 2018), we used a different approach, considering the problem from the point of view of local linearity by adapting a suggestion from David Tall (1992) that allowed us to place emphasis on the geometry of the problem. Thus, we focus on the notion of vertical change on a plane: $\Delta z = m_x \Delta x + m_y \Delta y$ (Figure 4), from an initial point (a, b, c) to a final generic point (x, y, z).

We used vertical change on a plane to consider the point-slopes equation for the plane, $z - c = m_x(x - a) + m_y(y - b)$, which in turn is used to describe the tangent plane to a bivariate function at a certain point: $z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$. Vertical change on a plane was also used to discuss the total differential of a function f at a point (a, b) as $df(a, b) = f_x(a, b)dx + f_y(a, b)dy$, and the directional derivative as $D_{\langle \Delta x, \Delta y \rangle} f(a, b) = \big(f_x(a, b)\Delta x +$

$f_y(a,b)\Delta y\big)/\|\langle\Delta x,\Delta y\rangle\|$. Note that the idea of vertical change on a plane can give coherence to the differential calculus Schema for two-variable function by interrelating several important notions of differential calculus in a way that makes them easier to access and use in problem situations.

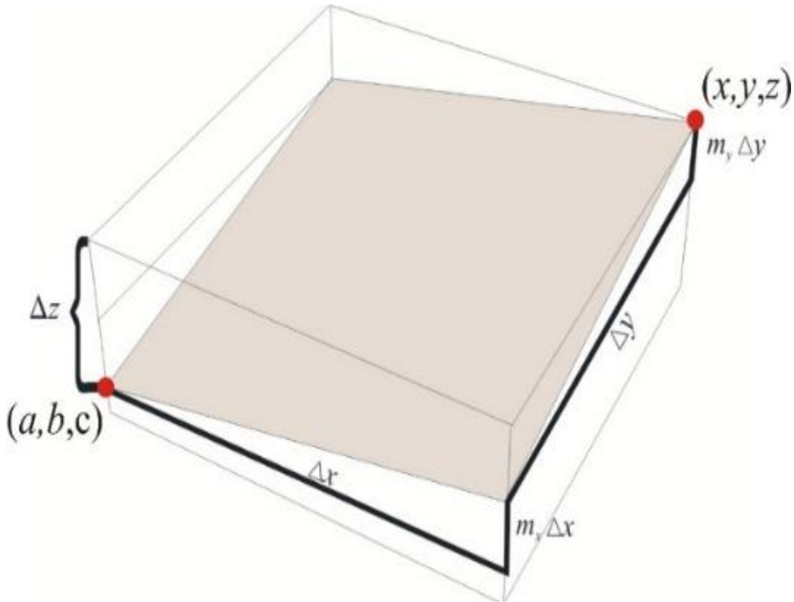

Figure 4 – Vertical change on a plane $\Delta z = m_x\Delta x + m_y\Delta y$.

3.A. First cycle of research: differential calculus of two-variable functions

Taking into account the aforementioned ideas on change in 3D space, we designed a genetic decomposition for the differential calculus of two-variable functions using vertical change on a plane as the main idea. Activities based on genetic decomposition were designed independently by two professors and used in two sections of a course on multivariate function calculus that they taught. They also used the ACE cycle as a didactic strategy. A research study was designed to test the GD (Martínez-Planell, Trigueros & McGee, 2015). The two APOS-based sections and one lecture-based section were included in the study to compare findings.

Results showed that the responses of students in the two sections that used the APOS methodology were quite similar, as both groups showed better learning results than the students in the lecture-based section. Most students in the lecture section were able to perform some of the Actions predicted by the DG model but the APOS students could carry out most of those Actions and some of the Processes predicted by the GD. Nevertheless, the analysis of students' responses related, for example, to the recognition that slope in the x or y direction on a non-vertical plane is independent of the initial point selected on the plane or to the coordination of vertical change in the x direction to that in the y direction before determining the total vertical change on a plane (Figure 4), included in the proposed GD, required more careful consideration in the proposed activities to help them reflect on the basic construction related to the tangent plane to a surface.

3.B. Partial derivatives

The construction of the partial derivatives Process involves coordinating two Processes, one for the derivative of one-variable function and one for a fundamental plane. Results revealed that several students in all three sections constructed an Action conception of the derivative of one-variable functions, which suggested the need to reconsider and reconstruct this topic when teaching the derivative of bivariate functions in 3D space. In addition, our results indicated that the construction of relations between different representations of partial derivatives seemed to be important for students as they sought to coordinate the Processes involved in constructing partial derivatives as a Process. Results also showed that students in both APOS sections more often evidenced the construction of the coordination needed to relate different representations of partial derivatives than those in the lecture-based section.

3.C. Directional derivative

Results of the interviews with students in all three sections raised the question of what they understood as the directional derivative. Most students in the lecture section referred to this as a formula, $(D_{\langle u_1,u_2\rangle}f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2)$, that few could recall, and even those that could were unable to explain its meaning geometrically. The students in the APOS sections also related it to a formula, now given as $D_{\langle \Delta x,\Delta y\rangle}f(a,b) = \left(f_x(a,b)\Delta x + f_y(a,b)\Delta y\right)/\|\langle \Delta x,\Delta y\rangle\|$. While about half of the students could remember and use this formula, when they were asked to give a geometrical justification or discuss the sign of a directional derivative on a graphical representation of a function, very few succeeded in giving an appropriate description that can be interpreted geometrically as in Figure 5. During the interviews, we inquired about the direction vector in 3D space, and found evidence that they had difficulty in representing it geometrically in space (Martínez-Planell, Trigueros, and McGee, 2017).

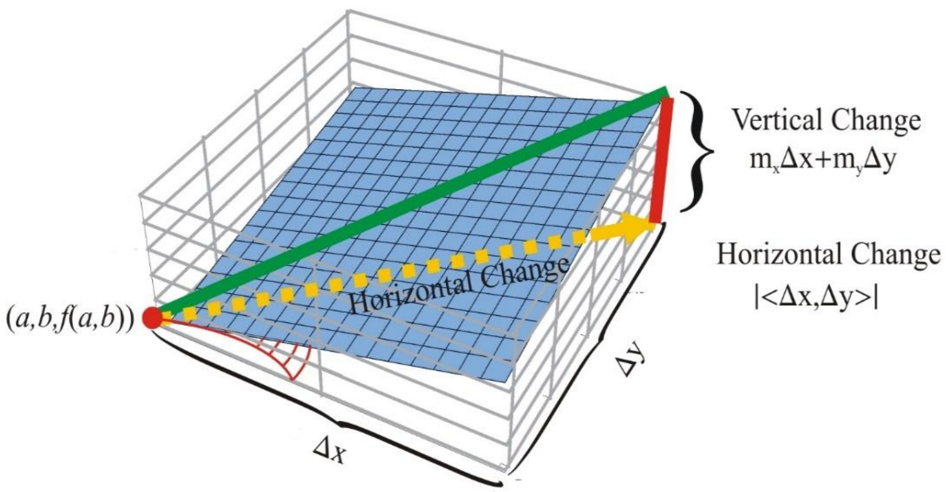

Figure 5 – Directional derivative at a point: $D_{\langle \Delta x,\Delta y\rangle}f(a,b) = \frac{f_x(a,b)\Delta x + f_y(a,b)\Delta y}{\|\langle \Delta x,\Delta y\rangle\|}$

3.D. Total differential

In our approach, using a tangent plane at a point, and a function in a local neighborhood of the point, the total differential can be related to the plane, as shown in Figure 6. Here, the total differential at a point gives the vertical change along the tangent plane as a function of the horizontal change (dx, dy). In our study, most students did not even remember what the total differential was. The few students who showed an Action conception of total differential were in the APOS sections. With only one exception, students could, at best, repeat memorized Actions, but were unable to justify them. For the most part, students did not relate the total differential at a point to the tangent plane at that point and the function in a neighborhood of the point. When they were helped to think about the expression, $df(a,b) = f_x(a,b)dx + f_y(a,b)dy$, they did not show evidence that they considered it as a function of the independent variables, dx, dy. They could only conceive of dx and dy as "very small numbers". Clearly, this suggested the need to revise the GD and APOS-based activities related to total differential (Martínez-Planell, Trigueros, and McGee, 2018).

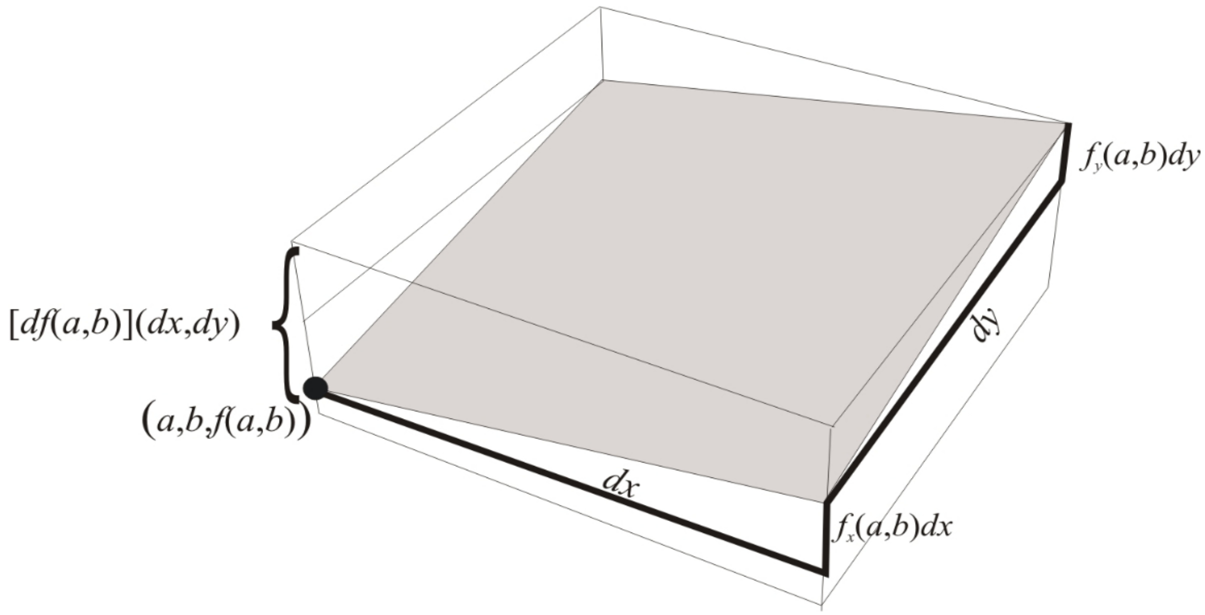

Figure 6 – Total differential at a point (a,b): $[df(a,b)](dx,dy) = f_x(a,b)dx + f_y(a,b)dy$.

When considering the overall findings on differential calculus we observed that while students in the APOS sections showed a better understanding than those in the lecture section, the results were not as good as expected. Students in the APOS sections were more likely to be able to perform Actions to compute vertical change on a plane and the point-slopes equation of a plane, and could better interpret partial derivatives geometrically and numerically, but in the case of directional derivatives or the total differential of functions their constructions did not differ significantly from those of the students in the lecture section. These results clearly indicated the need to refine the proposed GD based on the responses of participants in this study and to design new activities to be used and analyzed in a new research cycle on the topic of differential calculus.

To date, we have refined the GD and the activities but are just beginning to conduct a second research cycle. Since in the first research cycle we omitted some topics usually included in courses on differential calculus for two-variable functions, we are extending our research to include such elements as the conceptualization of partial derivatives and directional derivatives as Objects, second partial derivatives and mixed partial derivatives, the gradient vector, and the point-normal equation of a plane.

4. On students' ideas about integral calculus

There is scarce research on the learning and teaching of multivariable integral calculus. We began our work by asking: what constructions do students need to relate a double integral with its

Riemann sums?; and which constructions can be evidenced by students who have finished a lecture-based course on multivariable calculus?

Following the methodology of APOS theory, we began by designing a GD to explore students' basic understanding of Riemann sums. Then we designed a semi-structured interview instrument to explore students' conceptions of very simple Riemann sums related to bivariate functions considering a continuous two-variable function on a rectangular domain and the simplest possible partition; that is, the entire rectangle with no partitions, and a point in the rectangle (Figure 7). The goals were to examine several of the students' constructions described in the GD: recognizing the rectangle and function, forming a term of a Riemann sum, forming a more complex partition and a Riemann sum, and selecting a point in the rectangular domain to obtain a sub-estimation (Figure 7), an overestimation, and the exact value of the integral of this two-variable function. We also sought to determine how they conceived the relation of Riemann sums to the function's double integral.

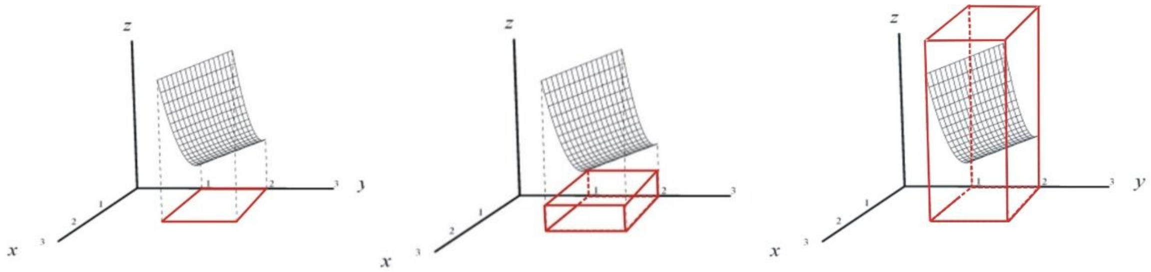

Figure 7 – Rectangle and function, sub-estimation, overestimation.

The students in this first research cycle were from a lecture section, so they had not used any APOS-based activities in their classroom, and their construction of rectangular domain and function over the rectangle was similar to that observed in the first round of geometric aspects and the definition of two-variable functions, discussed above. The analysis of their responses on the meaning of a Riemann sum term, $f(a,b)\Delta x\Delta y$, revealed that many had not yet constructed the meaning of this product, including some students who were able to perform Actions to interpret the meaning of each one of its components; that is, $f(a,b)$, Δx, Δy, and $\Delta x\Delta y$. Some students interpreted $f(a,b)$ as a location or point, such that $f(x,y)dxdy$ was the "area of a small piece of surface at a point", and the double integral $\iint f(x,y)dA$ was the surface area, rather than the volume between the rectangular domain and the surface. Some students also showed that they had not yet constructed the meaning of a double integral in a contextual situation, even though some could discuss the contextual units that corresponded to a term of a Riemann sum or to the expression $f(x,y)dxdy$ as an Action. Many confused the meanings of area and volume. These results suggested the need to explicitly discuss the various elements of this expression including the meaning of each factor $f(a,b)$, Δx, Δy, and $\Delta x\Delta y$ in a term of a Riemann sum, multiplying these factors to obtain and interpret a term of a Riemann sum, $f(a,b)\Delta x\Delta y$, including in contextual situations, forming more complex Riemann sums, understanding the existence of points where an underestimate, an overestimate, or the exact value of the double integral may be obtained in each sub-rectangle of a partition, and relating Riemann sums to their double integrals. Results further suggested the importance of relating verbal, numeric, and geometric representations of Riemann sums, and their algebraic representations as extended sums and as sums using sigma notation, before taking limits and representing the result as a double integral. The GD now contemplates paying explicit attention to this chain of representations as this has been shown to improve students' understanding in other studies (McGee & Martínez-Planell, 2014).

The analysis of the data and evidence gathered from the students' responses were useful in reviewing and refining the corresponding elements of the GD (Martínez-Planell & Trigueros, 2020). We have designed teaching activities that will soon be applied in the classroom to begin a second research cycle on the integration of bivariate functions. We also expect to extend the study to other elements of the GD that we have not yet tested, such as Riemann sums and their limits, how students work with problems that have more complex domains, and double and triple integrals when other coordinates –polar, cylindrical, spherical–are introduced.

5. Final remarks

Our results on the learning of bivariate functions clearly show that this process and the learning of the calculus associated with it require more than just generalizing the knowledge that students acquire when handling one-variable functions. Learning these topics involves reconstructing the calculus of one-variable function, and this reconstruction requires students to perform specific Actions on two-variable functions, their domain and range, and their graphs so that reflecting on them will allow them to construct Processes regarding their similarities to, and differences from, one-variable functions. Doing this, together with collaborative work among students and discussions with professors can help students understand this new type of functions in greater depth.

Using APOS theory as a framework helps researchers and professors identify the constructions that can play a fundamental role in students' learning, as they develop the reconstruction they need to fully understand these functions. Observing professors and analyzing textbooks allowed us to find evidence of a common assumption; namely, that students can form generalizations without much help. However, our research, and that of other authors using different theoretical frameworks for studies in other countries, such as the United States, Turkey, Iran, Malaysia, and Ethiopia, clearly shows that this is not the case (Martínez-Planell & Trigueros, 2021).

A solid construction of bivariate functions is indispensable for students to develop a thorough understanding and learn the notions involved in the differential and integral calculus of two-variable functions. The construction of these notions involves coordinating various Processes and constructing relations among distinct representation registers. It is important for professors to be aware of the fact that making an effort to help students carry out the necessary constructions of the geometric aspects and definition of two-variable function plays a key role in enhancing their learning of the notions involved in differential and integral calculus.

Our results address only a small portion of all the notions involved in the study of the learning and teaching of two-variable functions. Much additional research is needed for many areas of multivariable calculus not yet examined by researchers. Research on the teaching of bivariate and multivariate functions from different theoretical perspectives is needed, as are studies of the possibilities for teaching and learning that the use of technology in the classroom may open for students' learning. However, we strongly believe that results obtained from the research on bivariate functions conducted to date offer professors important ideas for approaching this topic from a richer perspective that can propose better possibilities for more students with different interests to learn these fundamental topics in a more significant manner.

References

Aguilar, M. S. (2020). Replication studies in mathematics education: What kind of questions would be productive to explore? *International Journal of Science and Mathematics Education*, *18*(1), 37-50. https://doi.org/10.1007/s10763-020-10069-7

Arnon, I., Cotrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS Theory: A framework for research and curriculum development in mathematics education.* Springer Verlag.

Borji, V., Martínez-Planell, R., & Trigueros, M. (2022). Student understanding of functions of two variables: A reproducibility study. *The Journal of Mathematical Behavior*, *66*, 100950. https://doi.org/10.1016/j.jmathb.2022.100950

Dubinsky, E., & McDonald, M. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 273–280). Kluwer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*(1), 103-131. https://doi.org/10.1007/s10649-006-0400-z

Harel, G., & Tall, D. (1989). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics*, *11*(1), 38-42.

McGee, D., & Martínez-Planell, R. (2014). A study of semiotic registers in the development of the definite integral of functions of two and three variables. *International Journal of Science and Mathematics Education*, *12*(4), 883-916.

Martínez-Planell, R. & Trigueros, M. (2012). Students' understanding of the general notion of a function of two variables. *Educational Studies in Mathematics*, *81*(3), 365-384. https://doi.org/10.1007/s10649-012-9408-8

Martínez-Planell, R. & Trigueros, M. (2013). Graphs of functions of two variables: Results from the design of instruction. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 663-672. https://doi.org/10.1080/0020739X.2013.780214

Martínez-Planell, R. & Trigueros, M. (2019). Using cycles of research in APOS: The case of functions of two variables. *The Journal of Mathematical Behavior*, *55*, Article 100687. https://doi.org/10.1016/j.jmathb.2019.01.003

Martínez-Planell, R. & Trigueros, M. (2020). Students' understanding of Riemann sums for integrals of functions of two variables. *The Journal of Mathematical Behavior*, *59*, 100791. https://doi.org/10.1016/j.jmathb.2020.100791

Martínez-Planell, R. & Trigueros, M. (2021). Multivariable calculus results in different countries. *ZDM-Mathematics Education 53*, 695–707. https://doi.org/10.1007/s11858-021-01233-6

Martínez-Planell, R., Trigueros, M. & McGee, D. (2015). On students' understanding of the differential calculus of functions of two variables. *The Journal of Mathematical Behavior*, *38*, 57-86. https://doi.org/10.1016/j.jmathb.2015.03.003

Martínez-Planell, R., Trigueros, M., & McGee, D. (2017). Students' understanding of the relation between tangent plane and directional derivatives of functions of two variables. *The Journal of Mathematical Behavior*, *46*, 13-41. https://doi.org/10.1016/j.jmathb.2017.02.001

Moore-Russo, D., Conner, A., & Rugg, K. (2011). Can slope be negative in 3-space? Studying concept image of slope through collective definition construction. *Educational Studies in Mathematics*, *76*(1), 3-21. https://doi.org/10.1007/s10649-010-9277-y

Piaget, J. (1970). L'épistémologie génétique. Presses universitaires de France.

Şefik, Ö & Dost, Ş. (2020). The analysis of the understanding of three-dimensional (Euclidian) space and the two-variable function concept by university students. *The Journal of Mathematical Behavior*, *57*, 100697. https://doi.org/10.1016/j.jmathb.2019.03.004

Tall, D. (1992). Visualizing differentials in two and three dimensions. *Teaching Mathematics and its Applications*, *11*(1), 1–7. http://dx.doi.org/10.1093/teamat/11.1.1

Trigueros, M., & Martínez-Planell, R. (2010). Geometrical representations in the learning of two-variable functions. *Educational Studies in Mathematics* *73*(1), 3-19. https://doi.org/10.1007/s10649-009-9201-5

Trigueros, M., & Martínez-Planell, R. (2015). Las funciones de dos variables: Análisis mediante los resultados del diálogo entre la teoría APOS y la TAD. *Enseñanza de las Ciencias*, *33*(2), 157-171. http://dx.doi.org/10.5565/rev/ensciencias.1520

Trigueros, M., Martínez-Planell, R., & McGee, D. (2018). Student understanding of the relation between tangent plane and the total differential of two-variable functions. *International Journal of Research in Undergraduate Mathematics Education*, *4*(1), 181-197. https://doi.org/10.1007/s40753-017-0062-5

Weber, E. D. (2015). The two-change problem and calculus students' thinking about direction and path. *The Journal of Mathematical Behavior*, *37*, 83-93. https://doi.org/10.1016/j.jmathb.2014.12.003

Yerushalmy, M. (1997). Designing representations: Reasoning about functions of two variables. *Journal of Research in Mathematics Education*, *28*, 431-466. https://doi.org/10.5951/jresematheduc.28.4.0431

María Trigueros
Instituto Tecnológico Autónomo de México
e-mail: mtriguerosg@gmail.com

Rafael Martínez-Planell
University of Puerto Rico at Mayagüez, Department of Mathematical Sciences
e-mail: rmplanell@gmail.com